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Double Absolute Matrix Summability Methods

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ABSTRACT

Matrix summability is one of the important summability methods. Different researchers have worked on $|A|_k$ summability of infinite series with real sequences. In this paper a new result on $|A, p_m, q_n|_k$ summability of doubly infinite lower triangular matrix has been established that generalizes a theorem of E. Savas and B.E. Rhoades on summability factor of double infinite weighted mean matrix.

MSC: 40F05; 40D15; 40G99

Keyword: Absolute Summability; Double summability; Double weighted mean; Summability factors; Infinite Series

I. INTRODUCTION

Let A be a lower triangular matrix.For any sequence (s_n) the n^{th} term of the A-transform of it is defined by

$$A_n = \sum_{v=0}^{n} a_{nv} s_v.$$

A series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty. \tag{1.1}$$

Let $A = (a_{mnjk})$ be a doubly infinite matrix. It is said to be doubly triangular if $a_{mnjk} = 0$ for $j > m$ or $k > n$. For any double sequence $\{s_{mn}\}$,the mn^{th} term of the A-transform of $\{s_{mn}\}$ is defined by

$$T_{mn} = \sum_{\mu=0}^{n} \sum_{v=0}^{n} a_{mn\mu v} s_{\mu v}.$$

For any double sequence $\{u_{mn}\}$, we define

$$\Delta_{11} u_{mn} = u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1}.$$

Similarly for any fourfold sequence v_{mnij}, we define

$$\Delta_{11} v_{mnij} = v_{mnij} - v_{m+1,n,i,j} - v_{m,n+1,i,j} + v_{m+1,n+1,i,j},$$

$$\Delta_{ij} v_{mnij} = v_{mnij} - v_{m,n,i+1,j} - v_{m,n,i,j+1} + v_{m,n,i+1,j+1},$$

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$\Delta_{0j} v_{mnij} = v_{mnij} - v_{m,n,i,j+1}$ and

$$\Delta_{i\ 0} v_{mnij} := v_{mnij} - v_{m,n,i+1,j}. \tag{1.2}$$

A double series $\sum\sum b_{mn}$, with sequence of partial sum $\{s_{mn}\}$ is said to be summable $|A|_k$, $k \geq 1$, if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\Delta_{11}T_{m-1,n-1}|^k < \infty. \tag{1.3}$$

We define the mn^{th} term of the double weighted mean transform of a double sequence $\{s_{mn}\}$ by

$$t_{mn} = \frac{1}{P_{mn}}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij}s_{ij},$$

where $$P_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij}$$

Further, a double infinite weighted mean matrix is said to be factorable[1], if there exist sequences $(p_m), (q_n)$ such that $p_{mn} = p_m q_n$ for every m and n.

A double series $\sum\sum b_{mn}$ is said to be summable $|\bar{N}, p_m, q_n|_k, k \geq 1$, if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|\Delta_{11}t_{m-1,n-1}|^k < \infty, \tag{1.4}$$

and the series $\sum\sum b_{mn}$ is summable $|A, p_m, q_n|_k, k \geq 1$, if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|\Delta_{11}T_{m-1,n-1}|^k < \infty, \tag{1.5}$$

Clearly, if we take $a_{mnij} = \frac{p_iq_j}{P_iQ_j}$, the $|A, p_m, q_n|_k$ summability is the same as $|\bar{N}, p_m, q_n|_k$ summability.

Associate with the matrix A, we consider two doubly triangular matrices $\bar{A}$ and $\hat{A}$ as follows:

$$\bar{a}_{mnij} = \sum_{\mu=i}^{m}\sum_{v=j}^{n} a_{mn\mu v},\ m, n, i, j = 0, 1, 2, ...$$

and

$$\hat{a}_{m,n,i,j} = \Delta_{11}\bar{a}_{m-1,n-1,i,j} \quad m, n = 0, 1, 2, ... \tag{1.6}$$

Note that $\hat{a}_{0000} = \bar{a}_{0000} = a_{0000}$.

Let y_{mn} denote the mn^{th} term of the A-transform of a factored doubly series $\sum_{\mu=0}^{m}\sum_{v=0}^{n} b_{\mu v\lambda_{\mu v}}$. Then we write

$$\begin{aligned} y_{mn} &= \sum_{\mu=0}^{m}\sum_{v=0}^{n} a_{mn\mu v}\sum_{i=0}^{\mu}\sum_{j=0}^{v} b_{ij}\lambda_{ij} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\sum_{\mu=i}^{m}\sum_{v=j}^{n} a_{mn\mu v} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\bar{a}_{mnij}. \end{aligned}$$

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Then we have

$$\begin{aligned}\Delta_{11}y_{m-1,n-1} &= y_{m-1,n-1} - y_{m,n-1} - y_{m-1,n} + y_{mn} \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} b_{ij}\lambda_{ij}\bar{a}_{m-1,n-1,i,j} - \sum_{i=0}^{m}\sum_{j=0}^{n-1} b_{ij}\lambda_{ij}\bar{a}_{m,n-1,i,j} \\ &- \sum_{i=0}^{m-1}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\bar{a}_{m-1,n,i,j} + \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\bar{a}_{mnij} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\hat{a}_{m,n,i,j} - \sum_{j=0}^{n-1} b_{mj}\lambda_{mj}\bar{a}_{m-1,n-1,m,j} \\ &- \sum_{i=0}^{m-1} b_{in}\lambda_{in}\bar{a}_{m-1,n-1,i,n} + \sum_{i=0}^{m} b_{in}\lambda_{in}\bar{a}_{m,n-1,i,n} + \sum_{j=0}^{n} b_{mn}\lambda_{mj}\bar{a}_{m-1,n,m,j} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\hat{a}_{mnij},\end{aligned}$$

since

$$\bar{a}_{m-1,n-1,m,j} = \bar{a}_{m-1,n-1,i,n} = \bar{a}_{m,n-1,i,n} = \bar{a}_{m-1,n,m,n} = 0$$

But as $b_{mn} = s_{m-1,n-1} - s_{m-1,n} - s_{m,n-1} + s_{mn}$,

$$\begin{aligned}\Delta_{11}y_{m-1,n-1} &= \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{mnij}\lambda_{ij}(s_{i-1,j-1} - s_{i-1,j} - s_{i,j-1} + s_{ij}) \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \hat{a}_{m,n,i+1,j+1}\lambda_{i+1,j+1}s_{ij} - \sum_{i=0}^{m-1}\sum_{j=0}^{n} \hat{a}_{m,n,i+1,j+1}\lambda_{i+1,j}s_{ij} \\ &- \sum_{i=0}^{m}\sum_{j=0}^{n-1} \hat{a}_{m,n,i,j+1}\lambda_{i,j+1}s_{ij} + \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{a}_{mnij}\lambda_{ij}s_{ij} \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \Delta_{ij}(\hat{a}_{mnij}\lambda_{ij})s_{ij} - \sum_{i=0}^{m-1} \hat{a}_{m,n,i+1,n}\lambda_{i+1,n}s_{in} \\ &- \sum_{j=0}^{n-1} \hat{a}_{m,n,m,j+1}\lambda_{m,j+1,n+1}s_{mj} + \sum_{i=0}^{n} \hat{a}_{mnmj}\lambda_{m,j}s_{mj} + \sum_{i=0}^{m-1} \hat{a}_{mnin}\lambda_{in}s_{in} \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \Delta_{ij}(\hat{a}_{mnij}\lambda_{ij})s_{ij} + \sum_{i=0}^{m-1} (\Delta_{i0}\hat{a}_{mnin}\lambda_{in})s_{in} \\ &+ \sum_{j=0}^{n-1}(\Delta_{0j}\hat{a}_{mnmj}\lambda_{mj})s_{mj} + \hat{a}_{mnmn}\lambda_{mn}s_{mn}. \qquad (1.7)\end{aligned}$$

Note that we may write

$$\Delta_{i0}\hat{a}_{mnin}\lambda_{in} = \lambda_{in}\Delta_{i0}\hat{a}_{mnin} + \hat{a}_{m,n,i+1,n}\Delta_{i0}\lambda_{in}$$

and

$$\Delta_{0j}\hat{a}_{mnmj}\lambda_{mj} = \lambda_{mj}\Delta_{0j}\hat{a}_{mnmj} + \hat{a}_{m,n,m,j+1}\Delta_{0j}\lambda_{mj},$$

so that

$$\begin{aligned}\sum_{i=0}^{m-1}(\Delta_{i0}\hat{a}_{mnin}\lambda_{in})s_{in} + \sum_{j=0}^{n-1}(\Delta_{0j}\hat{a}_{mnmj}\lambda_{mj})s_{mj} &= \sum_{i=0}^{m-1}[\lambda_{in}\Delta_{i0}\hat{a}_{mnin} + \hat{a}_{m,n,i+1,n}\Delta_{i0}\lambda_{in}]s_{in} \\ &+ \sum_{j=0}^{n-1}[\lambda_{mj}\Delta_{0j}\hat{a}_{mnmj} + \hat{a}_{m,n,m,j+1}\Delta_{0j}\lambda_{mj}]s_{mj}. \qquad (1.8)\end{aligned}$$

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It is easy to establish that for any two double sequences

$$\Delta_{ij}(u_{ij}v_{ij}) = v_{ij}\Delta_{ij}u_{ij} + (\Delta_{0j}u_{i+1,j})(\Delta_{i0}v_{ij}) + (\Delta_{i0}u_{i,j+1})(\Delta_{0j}v_{ij}) + u_{i+1,j+1}\Delta_{ij}v_{ij} \tag{1.9}$$

II. KNOWN RESULT

E. Savas and B.E. Rhoades [2] has proved the following result for $|\bar{N}, p_m, q_n|_k$ summability of double infinity series.

Theorem 1. Let $(p_m), (q_n)$ be sequence of positive numbers satisfying

(i) $P_mQ_n = O(mnp_mq_n)$ as $m, n \to \infty$,

Let X_{mn} be a given double sequence of positive numbers and suppose that $s_{mn} = O(X_{mn})$, as $m, n \to \infty$. If λ_{mn} is a double sequence of complex numbers satisfying

(ii) $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{p_mq_n}{P_mQ_n}(|\lambda_{mn}|X_{mn})^k = O(1)$,

(iii) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\Delta_{0j}\lambda_{ij}|X_{ij} = O(1)$,

(iv) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |\Delta_{i0}\lambda_{ij}|X_{ij} < \infty$,

(v) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\Delta_{ij}\lambda_{ij}|X_{ij} = O(1)$, and

(vi) $\sum_{i=0}^{m}\sum_{j=0}^{n} (|\lambda_{mn}|X_{mn})^k = O(1)$,

Then the series $\sum\sum b_{mn}\lambda_{mn}$ is summable $|\bar{N}, p_m, q_n|_k, k \geq 1$,

III. MAIN RESULT

The aim of this article is to generalize theorem-1 for double absolute factorable matrix summability.

Theorem 2. Let A be a doubly triangular matrix with non-negative entries satisfying the conditions

(i) $\Delta_{11}a_{m-1,n-1,i,j} \geq 0$

(ii) $\sum_{v=0}^{n} a_{mniv} = \sum_{v=0}^{n-1} a_{m,n-1,i,v} = b(m,i)$,

$\sum_{\mu=0}^{m} a_{mn\mu,j} = \sum_{\mu=0}^{m-1} a_{m-1,n,\mu,j} = a(n,j)$,

(iii) $a_{mnij} \geq max\{a_{m,n+1,i,j}a_{m+1,n,i,j}\}$ for $m \geq i, n \geq j$, and $i, j = 0, 1, ...,$

and

(iv) $\sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij} = O(1)$,

Let $\{X_{mn}\}$ be a given double sequence of positive numbers and suppose that $\{s_{mn}\} = O(X_{mn})$ as $m, n \to \infty$. If $\{\lambda_{mn}\}$ is a double sequence of complex numbers satisfying

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(v) $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mnmn}(|\lambda_{mn}|X_{mn})^k < \infty,$

(vi) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\Delta_{0j}\lambda_{ij}|X_{ij} = O(1),$

(vii) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |\Delta_{i0}\lambda_{ij}|X_{ij} < \infty,$

(viii) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\Delta_{ij}\lambda_{ij}|X_{ij} = O(1),$

and

(ix) $\sum_{i=0}^{m}\sum_{j=0}^{n}(|\lambda_{mn}|X_{mn})^k = O(1),$

Then the series $\sum\sum b_{mn}\lambda_{mn}$ is summable $|A, p_m, q_n|_k$, $k \geq 1$, where $(p_m), (q_n)$ are sequence of positive numbers such that

(x) $\sum_{m=1}^{n} p_m = P_n$ and $\sum_{m=1}^{n} q_m = Q_n$

and

(xi) $a_{mnmn} = O\left(\frac{p_m q_n}{P_m Q_n}\right)$

Proof. In order to prove the theorem, it is necessary, from (1.3), to show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{P_m Q_n}{p_m q_n}\right)^{k-1} |\Delta_{11} y_{mn}| < \infty,$$

From (1.9),

$$\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij}) = \lambda_{ij}\Delta_{ij}(\hat{a}_{mnij}) + (\Delta_{0j}\hat{a}_{m,n,i+1,j})(\Delta_{i0}\lambda_{ij}) + (\Delta_{i0}\hat{a}_{m,n,i,j+1})(\Delta_{0j}\lambda_{ij}) + \hat{a}_{m,n,i+1,j+1}\Delta_{ij}\lambda_{ij} \tag{3.1}$$

Using(3.1),

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij})s_{ij} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[\lambda_{ij}(\Delta_{ij}\hat{a}_{mnij}) + (\Delta_{0j}\hat{a}_{m,n,i+1,j})(\Delta_{i0}\lambda_{ij}) + (\Delta_{i0}\hat{a}_{m,n,i,j+1})(\Delta_{0j}\lambda_{ij}) + \hat{a}_{m,n,i+1,j+1}(\Delta_{ij}\lambda_{ij})]s_{ij}. \tag{3.2}$$

Therefore, using (1.7), (1.8) and (3.2), we may, write

$$\Delta_{11}y_{m-1,n-1} = \sum_{r=1}^{9} T_{mnr}.$$

Therefore, using (1.7), (1.8) and (3.2), we may, write

$$\Delta_{11}y_{m-1,n-1} = \sum_{r=1}^{9} T_{mnr}.$$

From Minkowski's inequality, it is sufficient to show that

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$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|T_{mnr}|^k<\infty, \text{ for } r=1,2,...,9.$$

Using Hölder's inequality,

$$I_1=\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|T_{mn1}|^k$$
$$=O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}||\lambda_{ij}|X_{ij}\right)^k$$
$$=O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}||\lambda_{ij}|^k|X_{ij}|^k\right)\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}|\right)^{k-1}$$

From (1.6),

$$\hat{a}_{mnij}=\Delta_{11}\bar{a}_{m-1,n-1,i,j}$$
$$=\bar{a}_{m-1,n-1,i,j}-\bar{a}_{m,n-1,i,j}-\bar{a}_{m-1,n,i,j}+\bar{a}_{mnij}$$
$$=\sum_{\mu=i}^{m-1}\sum_{v=j}^{n-1}a_{m-1,n-1,\mu,v}-\sum_{\mu=i}^{m}\sum_{v=j}^{n-1}a_{m,n-1,\mu,v}-\sum_{\mu=i}^{m-1}\sum_{v=j}^{n}a_{m-1,n,\mu,v}+\sum_{\mu=i}^{m}\sum_{v=j}^{n}a_{mn\mu v},$$

since $a_{m-1,n,m,v}=a_{m,n-1,\mu,n}=0$

Using (1.2) and property (ii)

$$\hat{a}_{mnij}=\sum_{\mu=i}^{m}\sum_{v=j}^{n}(a_{m-1,n-1,\mu,v}-a_{m,n-1,\mu,v}-a_{m-1,n,\mu,v}+a_{m,n,\mu,v})$$
$$=\sum_{\mu=i}^{m-1}[b(m-1,\mu)-\sum_{v=0}^{j-1}a_{m-1,n-1,\mu,v}-b(m,\mu)+\sum_{v=0}^{j-1}a_{m,n-1,\mu,v}$$
$$-b(m-1,\mu)+\sum_{v=0}^{j-1}a_{m-1,n,\mu,v}+b(m,\mu)-\sum_{v=0}^{j-1}a_{m,n,\mu,v}]$$
$$=\sum_{\mu=i}^{m-1}\sum_{v=j}^{n-1}(-a_{m-1,n-1,\mu,v}+a_{m,n-1\mu,v}+a_{m-1,n,\mu,v}-a_{m,n,\mu,v})$$
$$=\sum_{v=0}^{j-1}\sum_{\mu=i}^{m-1}(-a_{m-1,n-1,\mu,v}+a_{m,n-1\mu,v}+a_{m-1,n,\mu,v}-a_{m,n,\mu,v})$$
$$=\sum_{v=0}^{j-1}[-a(m-1,v)+\sum_{\mu=0}^{j-1}a_{m-1,n-1,\mu,v}+a(m,v)$$
$$-\sum_{\mu=0}^{i-1}a_{m,n-1,\mu,v}+a(m-1,v)-\sum_{\mu=0}^{i}a_{m-1,n,\mu,v}-a(m,v)+\sum_{\mu=0}^{i}a_{m,n,\mu,v}]$$
$$=\sum_{\mu=0}^{i-1}\sum_{v=0}^{j-1}\Delta_{11}a_{m-1,n-1,\mu,v}\geq 0. \tag{3.3}$$

Using (1.2) and (3.3),

$$\Delta_{ij}\hat{a}_{mnij}=\left(\sum_{\mu=0}^{i-1}\sum_{v=0}^{j-1}-\sum_{\mu=0}^{i}\sum_{v=0}^{j-1}-\sum_{\mu=0}^{i-1}\sum_{v=0}^{j}+\sum_{\mu=0}^{i}\sum_{v=0}^{j}\right)\Delta_{11}a_{m-1,n-1,\mu,v}$$
$$=-\sum_{v=0}^{j-1}\Delta_{11}a_{m-1,n-1,i,v}+\sum_{v=0}^{j}\Delta_{11}a_{m-1,n-1,i,v}$$
$$=\Delta_{11}a_{m-1,n-1,i,j}, \tag{3.4}$$

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and from condition (ii),

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}\hat{a}_{mnij} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(a_{m-1,n-1,i,j} - a_{m,n-1,i,j} - a_{m-1,n,i,j} + a_{mnij})$$
$$= \sum_{i=0}^{m-1}(b(m-1,i) - b(m,i) - b(m-1,i) + a_{m-1,n,i,n} + b(m,i) - a_{mnin})$$
$$= \sum_{i=0}^{m-1}(a_{m-1,n,i,n} - a_{mnin})$$
$$= a(n,n) - a(n,n) + a_{mnmn}.$$

Then

$$I_1 = O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}(|\lambda_{ij}|X_{ij})^k \sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}(a_{mnmn})^{k-1}|\Delta_{ij}\hat{a}_{mnij}|.$$
$$= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}(|\lambda_{ij}|X_{ij})^k \sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}|\Delta_{ij}\hat{a}_{mnij}|.$$

Using (3.4),

$$0 \le \sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}|\Delta_{ij}\hat{a}_{mnij}|$$
$$= \sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}(a_{m-1,n-1,i,j} - a_{m,n-1,i,j} - a_{m-1,n,i,j} + a_{mnij})$$
$$= \sum_{m=i+1}^{M+1}(a_{m-1,j,i,j} - a_{m-1,N+1,i,j} - a_{mjij} + a_{m,n+1,i.j})$$
$$= a_{ijij} - a_{M+1,j,i,j} - a_{i,N+1,i,j} + a_{M+1,N+1,i,j}$$
$$\le a_{ijij} \tag{3.5}$$

Hence, using condition(v),we get

$$I_1 = O(1)\sum_{i=0}^{M}\sum_{j=0}^{N}a_{ijij}(|\lambda_{ij}|X_{ij})^k = O(1).$$

Next,using Hölder's inequality,

$$I_2 = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|T_{mn2}|^k$$
$$= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left|\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(\Delta_{0j}\hat{a}_{m,n,i+1,j})(\Delta_{i0}\lambda_{ij})s_{ij}\right|^k$$
$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|\Delta_{i0}\lambda_{ij}|X_{ij}\right]$$
$$\times\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|\Delta_{i0}\lambda_{ij}|X_{ij}\right]^{k-1}$$

Using (3.3) and property (ii),

$$0 \le \hat{a}_{m,n,i+1,j} = \sum_{\mu=0}^{i}\sum_{v=0}^{j-1}\Delta_{11}a_{m-1,n-1,\mu,v}$$

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$$\le \sum_{\mu=0}^{m-1}\sum_{v=0}^{n-1}(a_{m-1,n-1,\mu,v} - _{m,n-1,\mu,v} - a_{m-1,n,\mu,v} + a_{m,n,\mu,v})$$
$$= \sum_{\mu=0}^{m-1}(b(m-1,\mu) - b(m,\mu) - b(m-1,\mu) + a_{m-1,n,\mu,n} + b(m,\mu) - a_{mn\mu v})$$
$$= \sum_{\mu=0}^{m-1}(a_{m-1,n,\mu,n} - a_{mn\mu v}).$$
$$= a(n,n) - a(n,n) + a_{mnmn}.$$

Since

$|\Delta_{0j}\hat{a}_{m,n,i+1,j}| \le \hat{a}_{m,n,i+1,j} + \hat{a}_{m,n,i+1,j+1}$,

using properties (vii),

$$= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}|\Delta_{i0}\lambda_{ij}|X_{ij}\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}(a_{mnmn})^{k-1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|.$$
$$= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}|\Delta_{i0}\lambda_{ij}|X_{ij}\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|.$$

From (2.3)

$$\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}| = O(1)\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}(\hat{a}_{m,n,i+1,j} + \hat{a}_{m,n,i+1,j+1})$$
$$= O(1)\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}\left(\sum_{\mu=0}^{i}\sum_{v=0}^{j-1}\Delta_{11}a_{m-1,n-1,\mu,v} + \sum_{\mu=0}^{i}\sum_{v=0}^{j}\Delta_{11}a_{m-1,n-1,\mu,v}\right)$$

Using conditions (i),(ii) and (iv),

$$\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}\sum_{\mu=0}^{i}\sum_{v=0}^{j-1}\Delta_{11}a_{m-1,n-1,\mu,v}$$
$$= \sum_{\mu=0}^{i}\sum_{v=0}^{j-1}\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}(a_{m-1,n-1,\mu,v} - a_{m,n-1,\mu,v} - a_{m-1,n,\mu,v} + a_{m,n,\mu,v})$$
$$= \sum_{\mu=0}^{i}\sum_{v=0}^{j-1}\sum_{m=i+1}^{M+1}(a_{m-1,j,\mu,v} - a_{m-1,N+1,\mu,v} - a_{m,j,\mu,v} + a_{m,N+1,\mu,v})$$
$$= \sum_{\mu=0}^{i}\sum_{v=0}^{j-1}(a_{i,j,\mu,v} - a_{M+1,j,\mu,v} - a_{i,N+1,\mu,v} + a_{M+1,N+1,\mu,v})$$
$$= \sum_{\mu=0}^{i}[b(i,\mu) - a_{i,j,\mu,j} - b(M+1,\mu) + a_{M+1,j,\mu,j} - b(i,\mu)$$
$$+ \sum_{v=j}^{N+1}a_{i,N+1,\mu,v} + b(M+1,\mu) - \sum_{v=j}^{N+1}a_{M+1,N+1,\mu,v}]$$

$$= \sum_{\mu=0}^{i}\left(-a_{i,j,\mu,j} + a_{M+1,j,\mu,j} + \sum_{v=j}^{N+1}(a_{i,N+1,\mu,v} - a_{M+1,N+1,\mu,v})\right)$$
$$= -a(j,j) + a(j,j) - \sum_{\mu=i+1}^{M+1}a_{M+1,j,\mu,j} + \sum_{v=j}^{N+1}\left(a(N+1,v) - a(N+1,v) + \sum_{\mu=i+1}^{M+1}a_{M+1,N+1,\mu,v}\right)$$
$$= \sum_{\mu=i+1}^{M+1}\sum_{v=j}^{N+1}a_{M+1,N+1,\mu,v} = O(1).$$

Similarly,

$$\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}\sum_{\mu=0}^{i}\sum_{v=0}^{j}\Delta_{11}a_{m-1,n-1,\mu,v}=O(1) \tag{3.6}$$

.

and hence $I_2 = O(1)$ by property (vii).

Similarly, we can prove that $I_3 = O(1)$.

Using Hölder's inequality,

$$\begin{aligned}
I_4 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|T_{mn4}|^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right)^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right]\times\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right]^{k-1}
\end{aligned}$$

From (2.3) and property (ii),

$$\begin{aligned}
0 \le \hat{a}_{m,n,i+1,j+1} &= \sum_{\mu=0}^{i}\sum_{v=0}^{j}\Delta_{11}a_{m-1,n-1,\mu,v} \\
&\le \sum_{\mu=0}^{m-1}\sum_{v=0}^{n-1}(a_{m-1,n-1,\mu,v} -_{m,n-1,\mu,v} - a_{m-1,n,\mu,v} + a_{m,n,\mu,v}) \\
&= \sum_{\mu=0}^{m-1}(b(m-1,\mu) - b(m,\mu) - b(m-1,\mu) + a_{m-1,n,\mu,n} + b(m,\mu) - a_{mn\mu v}) \\
&= \sum_{\mu=0}^{m-1}(a_{m-1,n,\mu,n} - a_{mn\mu v}). \\
&= a(n,n) - a(n,n) + a_{mnmn}.
\end{aligned}$$

Using properties (ii), (iv) and (viii),

$$\begin{aligned}
&= O(1)\sum_{i=0}^{M}\sum_{j=0}^{N}|\Delta_{ij}\lambda_{ij}|X_{ij}\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}(a_{mnmn})^{k-1}|\hat{a}_{m,n,i+1,j+1}| \\
&= O(1)\sum_{i=0}^{m-1}\sum_{j=0}^{N}|\Delta_{ij}\lambda_{ij}|X_{ij} \\
&= O(1).
\end{aligned}$$

Using (1.8) and Hölder's inequality,

$$\begin{aligned}
I_5 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|T_{mn5}|^k \\
&= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left|\sum_{i=0}^{m-1}\lambda_{in}\Delta_{i0}\hat{a}_{mnin}s_{in}\right|^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left(\sum_{i=0}^{m-1}\lambda_{in}|\Delta_{i0}\hat{a}_{mnin}|X_{in}\right)^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left[\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|(|\lambda_{in}|X_{in})^k\right]\times\left[\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|\right]^{k-1}
\end{aligned}$$

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From (1.6),

$$\Delta_{i0}\hat{a}_{mnin} = \Delta_{i0}(\Delta_{11}\bar{a}_{m-1,n-1,i,n})$$
$$= \Delta_{i0}(\bar{a}_{m-1,n-1,i,n} - \bar{a}_{m,n-1,i,n} - \bar{a}_{m-1,n,i,n} + \bar{a}_{mnin})$$
$$= \Delta_{i0}\left(-\sum_{\mu=i}^{m-1} a_{m-1,n,v,n} + \sum_{\mu=i}^{m} a_{mn\mu n}\right)$$
$$= -a_{m-1,n,i,n} + a_{mnin} \leq 0. \tag{3.7}$$

Using property (ii),

$$\sum_{i=0}^{m-1} |\Delta_{i0}\hat{a}_{mnin}| = \sum_{i=0}^{m-1}(a_{m-1,n-1,i,n} - a_{mnin})$$
$$= a(n,n) - a(n,n) + a_{mnmn}.$$
$$= O(1)\sum_{n=1}^{N+1}\sum_{i=0}^{M}(|\lambda_{in}|X_{in})^k\left(\sum_{m=i+1}^{M+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}(a_{mnmn})^{k-1}|\Delta_{i0}\hat{a}_{mnin}|\right)$$

From (2.7),

$$\sum_{m=i+1}^{M+1} |\Delta_{i0}\hat{a}_{mnin}| = \sum_{m=i+1}^{M+1}(a_{m-1,n,i,n} - a_{mnin})$$
$$= a_{inin} - a_{M+1,n,i,n} \leq a_{inin}.$$

Therefore,by property (vi),

$I_5 = O(1).$

Using Hölder's inequality

$$I_6 = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|T_{mn6}|^k$$
$$= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left|\sum_{i=0}^{m-1}\hat{a}_{m,n,i+1,n}(\Delta_{i0}\lambda_{in})s_{in}\right|^k$$
$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left(\sum_{i=0}^{m-1}|\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in}\right)^k$$
$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left[\sum_{i=0}^{m-1}|\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in}\right]\left[\sum_{i=0}^{m-1}|\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in}\right]^{k-1}$$

Using (1.6), and condition (ii),

$$\hat{a}_{m,n,i+1,n} = \bar{a}_{m-1,n-1,i+1,n} - \bar{a}_{m,n-1,i+1,n} - \bar{a}_{m-1,n,i+1,n} + \bar{a}_{m,n,i+1,n}$$
$$= -\sum_{\mu=i+1}^{m-1} a_{m-1,n,\mu,n} + \sum_{\mu=i+1}^{m} a_{m,n,\mu,n}$$
$$= -a(n,n) + \sum_{\mu=0}^{i} a_{m-1,n,\mu,n} + a(n,n) - \sum_{\mu=0}^{i} a_{m,n,\mu,n} \geq 0$$
$$\leq \sum_{\mu=0}^{m-1}(a_{m-1,n,\mu,n} - a_{m,n,\mu,n})$$
$$= a(n,n) - a(n,n) + a_{mnmn}.$$

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Thus, using condition (vii),

$$I_6 = O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}(a_{mnmn})^{k-1}\left[\sum_{i=0}^{m-1}|\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in}\right]\left[\sum_{i=0}^{m-1}|\Delta_{i0}\lambda_{in}|X_{in}\right]^{k-1}$$

$$= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N+1}|\Delta_{i0}\lambda_{in}|X_{in}\sum_{m=i+1}^{M+1}|\hat{a}_{m,n,i+1,n}|.$$

Using (3.3) and condition (ii),

$$\sum_{m=i+1}^{M+1}|\hat{a}_{m,n,i+1,n}| = \sum_{m=i+1}^{M+1}\sum_{\mu=0}^{i}(a_{m-1,n,\mu,n} - a_{m,n,\mu,n})$$

$$= \sum_{\mu=0}^{i}\sum_{m=i+1}^{M+1}(a_{m-1,n,\mu,n} - a_{m,n,\mu,n})$$

$$= \sum_{\mu=0}^{i}(a_{i,n,\mu,n} - a_{M+1,n,\mu,n}) \leq a(n,n) = O(1).$$

$$= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N+1}|\Delta_{i0}\lambda_{in}|X_{in}$$

$I_6 = O(1).$

Using Hölder's inequality,

$$I_7 = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|T_{mn7}|^k$$

$$= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left|\sum_{j=0}^{n-1}\lambda_{mj}(\Delta_{0j}\hat{a}_{mnmj})s_{mj}\right|^k$$

$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left(\sum_{j=0}^{n-1}|\lambda_{mj}||(\Delta_{0j}\hat{a}_{mnmj})|X_{mj}\right)^k$$

$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left[\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{mnmj}(|\lambda_{mj}||X_{mj})^k\right]\left[\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{mnmj}\right]^{k-1}$$

From (1.2),

$$\hat{a}_{mnmj} = \bar{a}_{m-1,n-1,m,j} - \bar{a}_{m,n-1,m,j} - \bar{a}_{m-1,n,m,j} + \bar{a}_{m,n,m,j}$$

$$= -\sum_{v=j}^{n-1}a_{m,n-1,m,j} + \sum_{v=j}^{n}a_{m,n,m,j}$$

Therefore

$$\Delta_{0j}\hat{a}_{mnmj} = -a_{m,n-1,m,j} + a_{m,m,m,j},$$

and using properties (ii) and (iii),

$$\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{mnmj}| = \sum_{j=0}^{n-1}(a_{m,n-1,m,j} - a_{m,n,m,j})$$

$$= b(m,m) - b(m,m) + a_{mnmn}.$$

Using properties (ix),

$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(|\lambda_{mj}|X_{mj})^k\sum_{n=j+1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}(a_{mnmn})^{k-1}|\Delta_{0j}\hat{a}_{mnmj}|$$

$= O(1).$

Using Hölder inequality,

$$I_8 = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|T_{mn8}|^k$$
$$= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left|\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}(\Delta_{0j}\lambda_{mj})s_{mj}\right|^k$$
$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left(\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}(\Delta_{0j}\lambda_{mj})X_{mj}\right)^k$$
$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}\left[\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}(\Delta_{0j}\lambda_{mj})X_{mj}\right]\left[\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}(\Delta_{0j}\lambda_{mj})X_{mj}\right]^{k-1}$$

Using an argument similar to that for the proof of I_6, and using properties (vi) we get

$$I_8 = O(1).$$

Finally using (1.7), properties (ii) and (v), and we that $\hat{a}_{mnmn} = a_{mnmn}$,

$$I_9 = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}|T_{mn9}|^k$$
$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{P_mQ_n}{p_mq_n}\right)^{k-1}(a_{mnmn})^{k-1}(a_{mnmn})(|\lambda_{mn}|X_{mn})^k$$
$$= O(1).$$

Which completes proof of theorem-2.

IV. CONCLUSION

If we take $p_m = 1$ and $q_n = 1$, then $|A, p_m, q_n|_k$ summability reduce to $|A|_k$ summability.

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