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LINEAR AND BILINEAR GENERATING FUNCTIONS OF HEAT TYPE POLYNOMIALS SUGGESTED BYJACOBI POLYNOMIALS

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ABSTRACT

The object of the manuscript is to obtain several types of linear and bilinear generating functions for the heat type polynomials suggested by Jacobi polynomials.

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Keywords:* *Jacobi polynomials, Appell's functions, Kampe'de Fe'riet's double hypergeometric function, Lauricella's functions, Gauss hypergeometric function.

I. INTRODUCTION

The present paper deals with an extension of earlier work **(**see*M.A. Khan, et al.*[1]).For the generalized heat polynomials $P_{n,v}(x,u)$ defined by (see *Haimo*[2], p. 736, eq. (2.1)).

$$P_{n,v}(x,u) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma\left(v+n+\frac{1}{2}\right)}{\Gamma\left(v+n-k+\frac{1}{2}\right)} x^{2n-2k} u^k \quad (1.1)$$

$$= (4u)^n n!\, L_n^{(v-\frac{1}{2})}\left(-\frac{x^2}{4u}\right) \quad (1.2)$$

The heat type polynomials suggested by Jacobi polynomials are denoted by, $P_{n,\lambda,\mu}(x,u)$ and defined as

$$P_{n,\lambda,\mu}(x,u) = (4u)^n \left(\lambda+\frac{1}{2}\right)_n \; {}_2F_1\left[\begin{matrix}-n, \lambda+\mu+n; & -\frac{x^2}{4u} \\ \lambda+\frac{1}{2} \quad ; & \end{matrix}\right] \quad (1.3)$$

$$= (4u)^n n!\; P_n^{\left(\lambda-\frac{1}{2},\mu-\frac{1}{2}\right)}\left(-\frac{x^2}{4\mathrm{u}}\right) \quad (1.4)$$

In the form similar to one given by *H.M.Srivastava* [3] for Jacobi polynomials (1.3) can be written as

$$P_{n,\lambda,\mu}(x,u) = \sum_{k=0}^{n} \binom{\lambda+n-\frac{1}{2}}{n-k}\binom{\mu+n-\frac{1}{2}}{k} n!\, x^{2k}(x^2+4u)^{n-k} \quad (1.5)$$

In view of the relation (see**,** E.D. *Rainville*[4], Th.20, pp.60), the relation (1.3) can be written in the elegant form

$$P_{n,\lambda,\mu}(x,u)=\left(\lambda+\frac{1}{2}\right)_n(x^2+4u)^n{}_2F_1\left[\begin{matrix}-n,\frac{1}{2}-\mu-n; & \dfrac{x^2}{x^2+4u}\\ \lambda+\frac{1}{2} \quad ; & \end{matrix}\right] \tag{1.6}$$

By means of the relation,

$$P_{n,\lambda,\mu}(\sqrt{x^2+4u}\,i,u)=(-1)^nP_{n,\mu,\lambda}(x,u)$$

which one can obtain by replacing x by $\sqrt{x^2+4u}\,i$ and t by $(-t)$ in our earlier work ([1], eq.(2.2)).

$$\sum_{n=0}^{\infty}\frac{P_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n\left(\mu+\frac{1}{2}\right)_n n!}t^n={}_0F_1\left[\begin{matrix}\ldots\ldots\ldots\ldots; & \\ \lambda+\frac{1}{2} \quad ; & x^2t\end{matrix}\right]\;{}_0F_1\left[\begin{matrix}\ldots\ldots\ldots; & \\ \mu+\frac{1}{2}; & (x^2+4u)t\end{matrix}\right] \tag{1.7}$$

another form of (1.3) is given by

$$P_{n,\lambda,\mu}(x,u)=(-1)^n(4u)^n\left(\mu+\frac{1}{2}\right)_{n\,2}F_1\left[\begin{matrix}-n,\lambda+\mu+n; & \dfrac{x^2+4u}{4u}\\ \mu+\frac{1}{2} \quad ; & \end{matrix}\right] \tag{1.8}$$

In finite series form (1.3), (1.6), and (1.8) can be written as,

$$P_{n,\lambda,\mu}(x,u)=\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\lambda+\frac{1}{2}\right)_n(\lambda+\mu)_{n+k}}{\left(\lambda+\frac{1}{2}\right)_k(\lambda+\mu)_n}(4u)^n\left(\frac{x^2}{4u}\right)^k \tag{1.9}$$

$$P_{n,\lambda,\mu}(x,u)=\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\lambda+\frac{1}{2}\right)_n\left(\mu+\frac{1}{2}\right)_n}{\left(\lambda+\frac{1}{2}\right)_k\left(\mu+\frac{1}{2}\right)_{n-k}}x^{2k}(x^2+4u)^{n-k} \tag{1.10}$$

$$P_{n,\lambda,\mu}(x,u)=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n+k}\left(\mu+\frac{1}{2}\right)_n(\lambda+\mu)_{n+k}}{\left(\mu+\frac{1}{2}\right)_k(\lambda+\mu)_n}\left(\frac{x^2+4u}{4u}\right)^k \tag{1.11}$$

Next we rewrite (1.9), (1.10) and (1.11) by reversing the order of summation in the form

$$P_{n,\lambda,\mu}(x,u)=\frac{(\lambda+\mu)_{2n}}{(\lambda+\mu)_n}x^{2n}{}_2F_1\left[\begin{matrix}-n,\frac{1}{2}-\lambda-n; & -\dfrac{4u}{x^2}\\ 1-\lambda-\mu-2n; & \end{matrix}\right] \tag{1.12}$$

$$P_{n,\lambda,\mu}(x,u)=\left(\mu+\frac{1}{2}\right)_n x^{2n}{}_2F_1\left[\begin{matrix}-n,\frac{1}{2}-\lambda-n; & \dfrac{x^2+4u}{x^2}\\ \mu+\frac{1}{2} \quad ; & \end{matrix}\right] \tag{1.13}$$

$$P_{n,\lambda,\mu}(x,u)=\frac{(\lambda+\mu)_{2n}}{(\lambda+\mu)_n}(x^2+4u)^n{}_2F_1\left[\begin{matrix}-n,\frac{1}{2}-\mu-n; & \dfrac{4u}{x^2+4u}\\ 1-\lambda-\mu-2n; & \end{matrix}\right] \tag{1.14}$$

Some of the definition and notations used in this paper are as follows:

Appell's four functions of two variables are given by [5].

$$F_1[a,b,b';c;x,y]=\sum_{n,k=0}^{\infty}\frac{(a)_{n+k}(b)_n(b')_k}{n!\,k!\,(c)_{n+k}}x^ny^k \tag{1.15}$$

$$F_2[a,b,b';c,c';x,y]=\sum_{n,k=0}^{\infty}\frac{(a)_{n+k}(b)_n(b')_k}{n!\,k!\,(c)_n(c')_k}x^ny^k \tag{1.16}$$

$$F_3[a,a',b,b';c;x,y]=\sum_{n,k=0}^{\infty}\frac{(a)_n(a')_k(b)_n(b')_k}{n!\,k!\,(c)_{n+k}}x^ny^k \tag{1.17}$$

$$F_4[a,b;c,c';x,y]=\sum_{n,k=0}^{\infty}\frac{(a)_{n+k}(b)_{n+k}}{n!\,k!\,(c)_n(c')_k}x^ny^k \tag{1.18}$$

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Further, Kampe′ de Fe′riet's type general double hypergeometric series (see *H.M. Srivastava,et al.* [6]) is defined as

$$F_{l:m;n}^{p:q;k}\left[\begin{matrix}(\alpha_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\tag{1.19}$$

Similarly, a general triple hypergeometric series $F^{(3)}[x,y,z]$ (see *H.M. Srivastava* [7], pp.428) is defined as

$$F^{(3)}[x,y,z]=F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{matrix}x,y,z\right]$$

$$=\sum_{m,n,p=0}^{\infty}\Lambda(m,n,p)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\tag{1.20}$$

where for convenience

$$\Lambda(m,n,p)=\frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}}$$

$$\times\frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p}\tag{1.21}$$

II. LINEAR GENERATING FUNCTIONS:

By using the relation (1.3), we obtain the certain generating function for $P_{n,\lambda,\mu}(x,u)$ is as follows:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n(\lambda+\mu)_nP_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n(\lambda+\mu-\alpha+1)_n(\lambda+\mu-\beta+1)_nn!}t^n$$

$$=(1-4ut)^{-\lambda-\mu}F_{2:1;0}^{2:2;1}\left[\begin{matrix}\frac{1}{2}(\lambda+\mu),\frac{1}{2}(\lambda+\mu+1):\alpha,\beta;\lambda+\mu-\alpha-\beta+1;\\\lambda+\mu-\alpha+1,\lambda+\mu-\beta+1:\lambda+\frac{1}{2};\ldots\ldots\ldots\ldots;\end{matrix}X,Y\right]\tag{2.1}$$

where,

$$X=\frac{4x^2t}{(1-4ut)^2},Y=\frac{-16ut}{(1-4ut)^2}\tag{2.2}$$

the limiting case of (2.1) when $|\beta|\to\infty$ yields

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\lambda+\mu)_nP_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n(\lambda+\mu-\alpha+1)_nn!}t^n$$

$$=(1-4ut)^{-\lambda-\mu}F_{1:1;0}^{1:2;1}\left[\begin{matrix}\frac{1}{2}(\lambda+\mu):\alpha,\frac{1}{2}(\lambda+\mu+1);\frac{1}{2}(\lambda+\mu+1)-\alpha;\\\lambda+\mu-\alpha+1:\alpha+\frac{1}{2};\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;\end{matrix}X,Y\right]\tag{2.3}$$

and

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\lambda+\mu)_nP_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n(\lambda+\mu-\alpha+1)_nn!}(-t)^n$$

$$=(1-4ut)^{-\lambda-\mu}F_{1:1;0}^{2:1;0}\left[\begin{matrix}\frac{1}{2}(\lambda+\mu),\frac{1}{2}(\lambda+\mu+1):\alpha;\ldots;\\\lambda+\mu-\alpha+1:\lambda+\frac{1}{2};\ldots\ldots;\end{matrix}-X,Y\right]\tag{2.4}$$

Now applying the definition (1.19) to rewrite $F_{1:1;0}^{2:1;0}$ as an infinite series of the Gaussian hypergeometric function

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$$ {}_2F_1\left[m+\frac{1}{2}(\lambda+\mu), m+\frac{1}{2}(\lambda+\mu+1); m+\lambda+\mu-\alpha+1; Y\right], (m=0,1,2,3,\ldots.) $$

and make use of Pfaff- Kummer transformation (see [8], pp.64)

$$ {}_2F_1[a,b;c;z] = (1-z)^{-a}\ {}_2F_1\left[a, c-b;\ c;\ \frac{z}{z-1}\right] \tag{2.5} $$

to each term of the series and replacing t on both sides of the resulting equation by $-t$, (2.4) thus leads us to the generating function (2.3).

A further limiting case of the generating function (2.3) when $|\alpha| \to \infty$ (or equivalently, of the generating function (2.1) when $\min\{|\alpha|, |\beta|\} \to \infty$) would similarly yields the result (see [1], eq.(2.1)),

$$ \sum_{n=0}^{\infty} \frac{(\lambda+\mu)_n P_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n n!} t^n = (1-4ut)^{-\lambda-\mu}\ {}_2F_1\left[\frac{1}{2}(\lambda+\mu), \frac{1}{2}(\lambda+\mu+1); \lambda+\frac{1}{2}; X\right] \tag{2.6} $$

where X is given by (2.2).

Relation (2.6) follows also when we set

$$ \alpha = \tfrac{1}{2}(\lambda+\mu+1) $$

in the generating function (2.3) or when we set

$$ \alpha = \beta = \tfrac{1}{2}(\lambda+\mu+1) $$

in the generating function (2.1). Also, the generating function (2.3) can be deduced from (2.1) by setting

$$ \beta = \tfrac{1}{2}(\lambda+\mu+1) $$

Alternative Derivation of the Generating Function(2.1):

By using one of many of special cases of his hypergeometric function (see[9], pp.76, eq.(3.1)), we thus obtain the generating function,

$$ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\gamma_j)_n P_{n,\lambda,\mu}(x,u)}{(\lambda+\frac{1}{2})_n (\mu+\frac{1}{2})_n \prod_{j=1}^{q} (\delta_j)_n n!} t^n = F^{p:0;0}_{q:1;1}\left[\begin{matrix}\gamma_1,\ldots,\gamma_p : \ldots\ldots ; \ldots\ldots ; \\ \delta_1,\ldots,\delta_q : \lambda+\frac{1}{2}; \mu+\frac{1}{2};\end{matrix}\ x^2 t, (x^2+4u)t\right] \tag{2.7} $$

Setting,

$$ \begin{cases} p=2=q=2, \gamma_1=\alpha, \gamma_2=\beta, \gamma_3=\lambda+\mu, \gamma_4=\mu+\dfrac{1}{2}, \\ \delta_1=\lambda+\mu-\alpha+1, \delta_2=\lambda+\mu-\beta+1, \end{cases} \tag{2.8} $$

the result (2.7) at once yields,

$$ \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\lambda+\mu)_n P_{n,\lambda,\mu}(x,u)}{\left(\lambda+\frac{1}{2}\right)_n (\lambda+\mu-\alpha+1)_n(\lambda+\mu-\beta+1)_n n!} t^n $$

$$ = F^{4:0;0}_{2:1;1}\left[\begin{matrix}\alpha,\ \beta,\ \lambda+\mu,\ \mu+\frac{1}{2} : \ldots\ldots\ldots ; \ldots\ldots\ldots ; \\ \lambda+\mu-\alpha+1, \lambda+\mu-\beta+1 : \lambda+\frac{1}{2}; \mu+\frac{1}{2};\end{matrix}\ x^2 t, (x^2+4u)t\right] \tag{2.9} $$

The left-hand sides of (2.1) and (2.9) are identical. In order to show their right-hand sides are also identical, let $\Omega(t)$ denote the second member of the generating function (2.9). Then it follows from the definition (1.19) and the identity (see[6], eq.142(312)).

$$ \sum_{m,n,p=0}^{\infty} A(m+n,p)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} = \sum_{n,p=0}^{\infty} A(n,p)\frac{(x+y)^n}{n!}\frac{z^p}{p!} \tag{2.10} $$

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that,

$$\Omega(t) = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m(\lambda+\mu)_m(\mu+\frac{1}{2})_m}{(\lambda+\mu-\alpha+1)_m(\lambda+\mu-\beta+1)_m(\lambda+\frac{1}{2})_m} \frac{(x^2t)^m}{m!}$$

$$\times \sum_{n,p=0}^{\infty} \frac{(\alpha+m)_{n+p}(\beta+m)_{n+p}(\lambda+\mu+m)_{n+p}(\mu+m+\frac{1}{2})_{n+p}}{(\lambda+\mu-\alpha+m+1)_{n+p}(\lambda+\mu-\beta+m+1)_{n+p}(\mu+\frac{1}{2})_{n+p}} \frac{(4ut)^n(x^2t)^p}{n!\,p!} \qquad (2.11)$$

Now replace the summation index m in (2.11) by $(m-p)$ and rearrange the resulting triple sum to the form:

$$\Omega(t) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}(\lambda+\mu)_{m+n}(\mu+n+\frac{1}{2})_m}{(\lambda+\mu-\alpha+1)_{m+n}(\lambda+\mu-\beta+1)_{m+n}(\lambda+\frac{1}{2})_m}$$

$$\times \ {}_2F_1\left[\begin{matrix}-m, -(\lambda-\frac{1}{2})-m; \\ \mu+n+\frac{1}{2} \ ;\end{matrix} 1\right] \frac{(x^2t)^m}{m!} \frac{(4ut)^n}{n!}$$

By applying Chu-Vandermonde theorem (see [10], pp.13) to sum the Gauss hypergeometric series with argument 1, we obtain

$$\Omega(t) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}(\lambda+\mu)_{2m+n}}{(\lambda+\mu-\alpha+1)_{m+n}(\lambda+\mu-\beta+1)_{m+n}(\lambda+\frac{1}{2})_m} \frac{(x^2t)^m}{m!} \frac{(4ut)^n}{n!} \qquad (2.12)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m(\lambda+\mu)_{2m}}{(\lambda+\mu-\alpha+1)_m(\lambda+\mu-\beta+1)_m(\lambda+\frac{1}{2})_m} \frac{(x^2t)^m}{m!}$$

$$\times \ {}_3F_2\left[\begin{matrix}\lambda+\mu+2m, \ \alpha+m, \ \beta+m; \\ \lambda+\mu-\alpha+m+1, \lambda+\mu-\beta+m+1;\end{matrix} 4ut\right]$$

Finally, by appealing to Whipple's transformation (see [10], pp.97, eq.4(iv)) and interpreting the double series by means of the definition (1.19), (2.12) leads the second member of generating function (2.1), which is further special case of the result (2.7).

Now we find some more extended linear generating function for the polynomials $P_{n,\lambda,\mu}(x,u)$ as follow

$$\sum_{n=0}^{\infty} \frac{(\lambda+\mu+m)_n P_{m+n,\lambda,\mu}(x,u)}{\left(\gamma+\frac{1}{2}\right)_n n!} t^n = \left(\lambda+\frac{1}{2}\right)_m (4u)^m \left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m}$$

$$\times F_4\left[\lambda+m+\frac{1}{2}, \lambda+\mu+m; \lambda+\frac{1}{2}, \gamma+\frac{1}{2}; \frac{x^2}{x^2+4u}, \frac{16u^2t}{x^2+4u}\right] \quad (2.13)$$

where F_4 denotes the fourth type of Appell's hypergeometric function of two variables defined by (1.18).

The generating relation (2.13) is an immediate consequence of the definition (1.6) and (1.18), and the familiar Gaussian hypergeometric transformation(see[11], eq.21, pp.33).

An interesting special case of the generating function (2.13) occurs when we set $m=0$ and $\gamma=\lambda$ and appealing the hypergeometric reduction formula,

$$F_4[\alpha+\beta+1, \alpha+1; \alpha+1, \alpha+1; x, y]$$

$$= (1-x-y)^{-\alpha-\beta-1} {}_2F_1\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2); \\ \alpha+1;\end{matrix} \frac{4xy}{(1-x-y)^2}\right] \qquad (2.14)$$

We thus arrive to the generating function(see *M.A. Khan et al.*[1], eq.(2.1)).

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Another extended generating function are obtained by replacing t by $(t+v)$ and making use of Binomial theorem on the left-hand side, and expanding the Gaussian function on right-hand side in the generating function(see*M.A. Khan et al.*[1], eq.(2.25)),

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{n,\lambda-n,\mu-n}(x,u)}{(1-\lambda-\mu)_n n!} t^n = (1+x^2t)^{-\alpha} {}_2F_1 \left[\begin{matrix} \alpha, -\left(\lambda-\frac{1}{2}\right); \\ 1-\lambda-\mu; \end{matrix} \frac{-4ut}{1+x^2t} \right] \quad (2.15)$$

which readily gives the relation in the elegant form

$$\sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} P_{m+n,\lambda-m-n,\mu-m-n}(x,u)}{(1-\lambda-\mu)_{m+n} n!} t^n (v)^m = \sum_{m=0}^{\infty} (-1)^m (\alpha)_m (x)^{2m} (1+x^2t)^{-\alpha-m}$$

$$\times F_1 \left[-\left(\lambda-\frac{1}{2}\right), -m, \alpha+m; 1-\lambda-\mu; -\frac{4u}{x^2}, -\frac{4ut}{1+x^2t} \right] (v)^m \quad (2.16)$$

Now equate the coefficient of $(v)^m$ in (2.16) and on replacing λ, μ and α by $\lambda+m, \mu+m, \alpha-m,$ we thus arrive to the generating function,

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{m+n,\lambda-n,\mu-n}(x,u)}{(1-\lambda-\mu-m)_n n!} t^n = (\lambda+\mu+m)_m (x)^{2m} (1+x^2t)^{-\alpha}$$

$$\times F_1 \left[-\left(\lambda-\frac{1}{2}\right) - m, -m, \alpha; 1-\lambda-\mu-2m; -\frac{4u}{x^2}, -\frac{4ut}{1+x^2t} \right] (2.17)$$

Similarly, replace t by $(t+v)$ in the generating function (see[1], eq.(2.26)),

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{n,\lambda-n,\mu-n}(x,u)}{(1-\lambda-\mu)_n n!} t^n = (1+(x^2+4u)t)^{-\alpha} {}_2F_1 \left[\begin{matrix} \alpha, -\left(\mu-\frac{1}{2}\right); \\ 1-\lambda-\mu; \end{matrix} \frac{4ut}{1+(x^2+4u)t} \right] (2.18)$$

and make use of Binomial theorem, we obtain

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{m+n,\lambda-n,\mu-n}(x,u)}{(1-\lambda-\mu-m)_n n!} t^n = (\lambda+\mu+m)_m (x^2+4u)^m (1+(x^2+4u)t)^{-\alpha}$$

$$\times F_1 \left[-\left(\mu-\frac{1}{2}\right) - m, -m, \alpha; 1-\lambda-\mu-2m; \frac{4u}{x^2+4u}, \frac{4ut}{1+(x^2+4u)t} \right] \quad (2.19)$$

By using $\alpha = (1-\lambda-\mu-m)$ in (2.19) and in conjunction with the hypergeometric reduction formula (see, *Erde'lyi et al.* [9], vol.I, pp.238, eq.(1)).

$$F_1[a, b, b'; b+b'; x, y] = (1-y)^{-a} {}_2F_1 \left[\begin{matrix} a, b; \\ b+b'; \end{matrix} \frac{x-y}{1-y} \right] \quad (2.20)$$

a new generating relation is obtained as follows:

$$\sum_{n=0}^{\infty} \frac{P_{m+n,\lambda-n,\mu-n}(x,u)}{n!} t^n = [1+(x^2+u)t]^{\lambda-\frac{1}{2}} [1+x^2t]^{\mu-\frac{1}{2}} P_{m,\lambda,\mu}[x^2(1+(x^2+4u)t), u] \quad (2.21)$$

If we rewrite (1.14) as

$$P_{m+n,\lambda-n,\mu-n}(x,u) = \frac{(\lambda+\mu-2n)_{2(m+n)}}{(\lambda+\mu-2n)_{m+n}} (x^2+4u)^{m+n} {}_2F_1 \left[\begin{matrix} -m-n, \frac{1}{2}-\mu-m; \\ 1-\lambda-\mu-2m; \end{matrix} \frac{4u}{x^2+4u} \right] (2.22)$$

and the Pfaff-Kummer's transformation (2.5), we thus arrive to an elegant form of generating function

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n P_{m+n,\lambda-n,\mu-n}(x,u)}{(\beta)_n n!} t^n = (\lambda+\mu+m)_m x^{2m} \left(\frac{x^2}{x^2+4u} \right)^{\mu-\frac{1}{2}}$$

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$$\times F_2\left[1-\lambda-\mu-m,-\left(\mu-\frac{1}{2}\right)-m,\alpha;1-\lambda-\mu-2m,\beta;\frac{-4u}{x^2},-(x^2+4u)t\right] \quad (2.23)$$

Similarly, from (1.12) we obtain

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n P_{m+n,\lambda-n,\mu-n}(x,u)}{(\beta)_n n!}t^n=(\lambda+\mu+m)_m(x^2+4u)^m\left(\frac{x^2+4u}{x^2}\right)^{\lambda-\frac{1}{2}}$$

$$\times \boldsymbol{F_2\left[1-\lambda-\mu-m,-\left(\lambda-\frac{1}{2}\right)-m,\alpha;1-\lambda-\mu-2m,\beta;\frac{4u}{x^2+4u},-x^2t\right]} \quad \mathbf{(2.24)}$$

III. BILINEAR GENERATING FUNCTIONS:

The polynomials $P_{n,\lambda,\mu}(x,u)$ admits the following generating functions involving the sets

$$\{P_{m+n,\lambda,\mu}(x,u).P_{n,\gamma,\delta}(y,u)\}_{n=0}^{\infty},$$

and

$$\{P_{n,\lambda-n,\mu-n}(x,u).P_{n,\gamma-n,\delta-n}(y,u)\}_{n=0}^{\infty}$$

where m is a non-negative integer,and the parameter $\lambda,\mu,\gamma,\delta$ are independent on n.

In view of the definition (1.10) and the generating function (2.13) would readily yields the generating function

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu+m)_n P_{m+n,\lambda,\mu}(x,u)P_{n,\gamma,\delta}(y,u)}{\left(\gamma+\frac{1}{2}\right)_n\left(\delta+\frac{1}{2}\right)_n n!}t^n=\left(\lambda+\frac{1}{2}\right)_m(4u)^m\left(\frac{x^2+4u}{4u}\right)^{-\lambda-\mu-m}$$

$$\times F_C^{(3)}\left[\lambda+\mu+m,\lambda+m+\frac{1}{2};\lambda+\frac{1}{2},\gamma+\frac{1}{2},\delta+\frac{1}{2};\frac{x^2}{x^2+4u},\frac{16y^2u^2t}{x^2+4u},\frac{16(y^2+4u)u^2t}{x^2+4u}\right] \quad (3.1)$$

where $F_C^{(3)}$ denote the Lauricella's function defined by**(**[11], eq.1.7(3)) with $n=3$.

In (3.1) set $m=0,\gamma=\lambda,\delta=\mu,$ and appealing the hypergeometric reduction formula (see**,***B.L.Sharma***[**14], pp.716, eq.(2.4))

$$F_C^{(3)}[\alpha+\beta+1,\beta+1;\alpha+1,\beta+1,\beta+1;x,y,z]$$

$$=(1+x-y-z)^{-\alpha-\beta-1}F_4\left[\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha+\beta+2);\alpha+1,\beta+1;X,Y\right] \quad (3.2)$$

where,

$$X=\frac{4x}{(1+x-y-z)^2},Y=\frac{4yz}{(1+x-y-z)^2}$$

leads the generating relation

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu)_n P_{n,\lambda,\mu}(x,u)P_{n,\lambda,\mu}(y,u)}{\left(\lambda+\frac{1}{2}\right)_n\left(\mu+\frac{1}{2}\right)_n n!}t^n$$

$$=(1+16u^2t)^{-\lambda-\mu}F_4\left[\frac{1}{2}(\lambda+\mu),\frac{1}{2}(\lambda+\mu+1);\mu+\frac{1}{2},\lambda+\frac{1}{2};X,Y\right] \quad (3.3)$$

where,

$$X=\frac{4(x^2+4u)(y^2+4u)t}{(1+16u^2t)^2},Y=\frac{4x^2y^2t}{(1+16u^2t)^2}$$

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If we rewrite (1.14) as

$$P_{n,\lambda-n,\mu-n}(x,u) = \frac{(\lambda+\mu-2n)_{2n}}{(\lambda+\mu-2n)_n}(x^2+4u)^n\,{}_2F_1\left[\begin{matrix}-n,\frac{1}{2}-\mu; & \frac{4u}{x^2+4u}\\ 1-\lambda-\mu; & \end{matrix}\right] \quad (3.4)$$

which in conjunction with (2.17) would leads at once the generating function

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n P_{n,\lambda-n,\mu-n}(x,u).P_{n,\gamma-n,\delta-n}(y,u)}{(1-\lambda-\mu)_n(1-\gamma-\delta)_n n!}t^n$$

$$= (1+x^2(y^2+4u)t)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n\left(-(\delta-\frac{1}{2})\right)_n\left(-(\lambda-\frac{1}{2})\right)_n}{(1-\gamma-\delta)_n(1-\lambda-\mu)_n n!}(\zeta)^n$$

$$\times F_2\left[\alpha+n,-\left(\delta-\frac{1}{2}\right)+n,-\left(\lambda-\frac{1}{2}\right)+n;1-\gamma-\delta+n,1-\lambda-\mu+n;\xi,\eta\right] \quad (3.5)$$

where,

$$\frac{\xi}{\frac{x^2}{4u}} = -\frac{\eta}{\frac{y^2+4u}{4u}} = \zeta = \frac{16u^2t}{1+x^2(y^2+4u)t}$$

The second member of (3.5) can indeed be written in terms of Srivastava triple hypergeometric series $F^{(3)}[x,y,z]$ defined by (1.20), and we thus obtain the alternative form of the bilinear generating function (3.5) as,

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n P_{n,\lambda-n,\mu-n}(x,u).P_{n,\gamma-n,\delta-n}(y,u)}{(1-\lambda-\mu)_n(1-\gamma-\delta)_n n!}t^n$$

$$= (1+x^2(y^2+4u)t)^{-\alpha}F^{(3)}\left[\begin{matrix}\alpha::-\left(\delta-\frac{1}{2}\right);-\left(\lambda-\frac{1}{2}\right):\ldots;\ldots;\ldots; \\ \ldots::1-\gamma-\delta;1-\lambda-\mu:\ldots;\ldots;\ldots;\end{matrix}\ \xi,\eta,\zeta\right] \quad (3.6)$$

Another interesting special case of (3.6) would occurs, when we set $\alpha = (1-\lambda-\mu)$, along with([5], pp.35, eq.10).

$$F_2[a,b,b';a,c';x,y] = (1-x)^{-b}F_1\left[b',b,a-b;c';\frac{y}{1-x},y\right] \quad (3.7)$$

and the power series identity(*H.M. Srivastava, et al.*[11], 1.6(2)).

We thus obtain the generating function

$$\sum_{n=0}^{\infty}\frac{P_{n,\lambda-n,\mu-n}(x,u).P_{n,\gamma-n,\delta-n}(y,u)}{(1-\gamma-\delta)_n n!}t^n$$

$$= (\theta)^{\lambda-\frac{1}{2}}(\phi)^{\mu-\frac{1}{2}}F_1\left[-\left(\delta-\frac{1}{2}\right),-\left(\lambda-\frac{1}{2}\right),-\left(\mu-\frac{1}{2}\right);1-\gamma-\delta;\frac{4u(x^2+4u)t}{\theta},\frac{4ux^2t}{\phi}\right] \quad (3.8)$$

where,

$$\theta = 1+(x^2+4u)(y^2+4u)t,\ \phi = 1+x^2(y^2+4u)t$$

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