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A NECESSARY AND SUFFICIENT CONDITION FOR A SIGNED GRAPH TO BE A REGULAR SIGNED GRAPH

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ABSTRACT

A signed graph is a graph S = (G,σ), where G = (V,E) in which each edge is assigned a positive or negative sign. Let S = (G,σ) be a signed graph with V= (v_1, v_2,..., v_n). In this paper we obtain the necessary and sufficient condition for a signed graph to be a regular signed graph.

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I. INTRODUCTION

A signed graph is defined to be a pair S = (G,σ), where G = (V,E) is the underlying graph and σ : E → {−1,1} is the signing function. These were first introduced by Harary [3]. The sets of positive and negative edges of S are respectively denoted by E^+ and E^-. Thus E=E^+∪ E^-. Our signed graphs have simple underlying graphs. A signed graph is said to be homogeneous if all of its edges have either positive sign or negative sign and heterogeneous, otherwise. A graph can be considered to be a homogeneous signed graph with each edge positive; thus signed graphs become a generalization of graphs. The sign of a signed graph is defined as the product of signs of its edges. A signed graph is said to be positive (respectively, negative) if its sign is positive (respectively, negative) i.e., it contains an even (respectively, odd) number of negative edges. A signed graph is said to be all-positive (respectively, all-negative) if all of its edges are positive (respectively, negative). A signed graph is said to be balanced if each of its cycles is positive and unbalanced, otherwise. We denote by −S the signed graph obtained by negating each edge of S and call it the negative of S. We call balanced cycle a positive cycle and an unbalanced cycle a negative cycle and respectively denote them by Cn and C_n, where, n is number of vertices.

The signed degree of v_i is sdeg(v_i) = d_i = ${d_i}^+$ + ${d_i}^-$, where 1 ≤ i ≤ n and ${d_i}^+$ (${d_i}^-$) is the number of positive(negative) edges incident with v_i. So the sequence σ = (d_1, d_2,..., d_n) in non-increasing order is called signed degree sequence of G. We denote positive edge xy by xy+ and a negative edge xy by xy−. An integral

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sequence is s-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence σ = (d_1, $d_2,\ldots, d_n$) is a standard sequence if σ is non-increasing, $\sum_{i=1}^{n} d_i$ is even, d_1>0, each $|d_i|$ < n and $|d_i| \geq |d_n|$.

Definition 1.1.

The signed star graph denoted by $K_{1,n}$ is the signed bipartite graph with partite sets 1 and n. The following result, due to Chartrand et al.[1], gives a necessary and sufficient condition for an integral sequence to be s-graphical, which is similar to Hakimi's result for graphical sequences in graphs [2].

Theorem 1.2. A standard integral sequence σ = (d_1, $d_2,\ldots, d_n$) is s-graphical if and only if $\sigma^{/}$ = (d_2-1,d_3-1, ⋯ , $d_{d_{1+s+1}}$,$d_{d_{1+s+2}}$,d_{n-s}, $d_{n-s+1}+1$,⋯ ,d_n+1) is s-graphical for some s,$0 \leq s \leq n-1-d_1$.

The next characterization for signed graphical sequences in signed graphs is given by Yan et al. [7]

Theorem 1.3. A standard integral sequence σ = (d_1, $d_2,\ldots, d_n$)is s-graphical if and only if $\sigma_m{}^{/}$ = (d_2-1, d_3-1, · · · , $d_{d_{1+s+1}}$, $d_{d_{1+s+2}}$,d_{n-m}, $d_{n-m+1}+1$,· · · ,d_n+1) is s-graphical where m is the maximum non-negative integer such that , $d_{d_{1+m+1}}>d_{n-m+1}$.

The set of distinct signed degrees of the vertices in a signed graph is called its signed degree set. Pirzada et al. [4] proved that every non-empty set of positive(negative) integers is the signed degree set of some connected signed graph and determine the smallest possible order for such a signed graph. In 2015, Pirzada et al. [6] obtained graphical sequences of some family of induced subgraphs and obtained new condition for a graphic sequence to be potentially K4 −e graphic. A complete signed bipartite graph with r vertices in one partite set and s vertices in another set is denoted byKr,s is a complete bipartite graph in which each edge is assigned a positive or a negative sign. A complete signed bipartite graph of the form K1,n−1 is called a star signed graph.

In 2007 Pirzada et al. [5] proved the following assertions in signed bipartite graphs.

Theorem 1.4.Let S(U,V) be a signed bipartite graph with m edges. Then,

$g = \sum_{u \in V}^{n} s\deg(u) = \sum_{v \in V}^{n} s\deg(v) = m(\bmod 2)$ and the number of positive edges and negative edges of G(U,V) are respectively (m+g)/ 2 and (m−g)/ 2 .

The following result gives a necessary and sufficient condition for a pair of integral sequences to be the signed degree sequences of some complete signed bipartite graph.

Theorem 1.5. Let α = (d_1, $d_2,\ldots, d_p$) and β = (e_1, $e_2,\ldots, e_q$) be standard sequences and let r = d_1+q_2 . Let α0 be obtained from α by deleting d_1 and β0 be obtained from β by reducing r greatest entries of β by 1 each and adding remaining entries of β by 1 each. Then α and β are the signed graphic sequences of some complete signed bipartite graph if and only if α0 and β0 are also.

II MAIN RESULTS

The following result gives the necessary and sufficient condition for a signed graph to be a regular signed graph.

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Theorem 2.1. A Signed graph S = (V,ζ) is regular if and only if $|\zeta|^2 = n/4 \sum_{i=1}^{n} |(sdeg v_i)|^2$.

Proof. Suppose that the signed graph S = (V,ζ) is a regular of degree r. Therefore 2|ζ| = nr and $sdeg v_i = r \; \forall \; i = 1,2,\cdots,n$. We know that,

$$(s\deg v_1)^2 + (s\deg v_2)^2 \ldots (s\deg v_n)^2 = (d_1^+ - d_1^-)^2 + (d_2^+ - d_2^-)^2 + \ldots + (d_n^+ - d_n^-)^2$$
$$= n(d_1^+ - d_1^-)^2 = nr^2.$$

Further, we have

$4|\zeta|^2 /n = 4/n \,.\, n^2 r^2 /4 = n\, r^2$ (1)

Therefore, from above we have

$$(s\deg v_1)^2 + (s\deg v_2)^2 \ldots (s\deg v_n)^2 = 4|\xi|^2 n$$
$$\Rightarrow |\xi|^2 = \frac{n}{4}\sum_{i=1}^{n}(s\deg v_i)^2.$$

Conversely, suppose that $|\xi|^2 = \frac{n}{4}\sum_{i=1}^{n}(s\deg v_i)^2$

$$\Rightarrow \frac{n}{4}\{(s\deg v_1)^2 + (s\deg v_2)^2 \ldots (s\deg v_n)\}^2 - |\xi|^2 = 0$$
$$\Rightarrow (s\deg v_1)^2 + (s\deg v_2)^2 \ldots (s\deg v_n)^2 - {}^1\!/_n |2\xi|^2 = 0$$

$$\Rightarrow (s\deg v_1)^2 + (s\deg v_2)^2 \ldots (s\deg v_n)^2 - \frac{1}{n}\{(s\deg v_1)^2 + (s\deg v_2)^2 \ldots (s\deg v_n)^2$$
$$+ 2(s\deg v_1\, s\deg v_2 + s\deg v_1\, s\deg v_3 \ldots + s\deg v_1\, s\deg v_n)$$
$$+ 2(s\deg v_2\, s\deg v_3 + s\deg v_2\, s\deg v_4 \ldots + s\deg v_2\, s\deg v_n) + 2(s\deg v_{n-1} s\deg v_n) = 0$$
$$\Rightarrow \frac{1}{n}(s\deg v_1 - s\deg v_2)^2 + \ldots + (s\deg v_1 - s\deg v_n)^2 + (s\deg v_2 - s\deg v_3)^2 + \ldots (s\deg v_2 - s\deg v_n)^2$$
$$+ (s\deg v_{n-1} - s\deg v_n)^2 = 0.$$

From this equation each term is non-negative for every i, j. Therefore, we know that it is possible only when $sdeg v_i = sdeg v_j$ for every i, j and therefore signed graph S = (V,ζ) is a regular signed graph. Hence completes the proof.

The following result gives the condition for a signed graph to be a complete signed graph.

Theorem 2.2. Let S be a signed graph with $n \geq 2$ vertices. Then S is a complete signed graph K_n if and only if

$$|\xi|^2 = \frac{n}{4}\sum_{i=1}^{n}(s\deg v_i)^2 = |\xi|^2\left\{2\xi/_{n-1} + n - 2\right\}$$

Proof. We know that a signed graph S is complete iff |ζ| = n(n−1) /2

⇔ 2|ζ| =n(n−1)

⇔2|ζ|(n−2) = n(n−1)(n−2)

⇔2|ζ|(n−2) + 2|ξ|n = 2|ζ|n + n(n−1)(n−2)

⇔4$|\zeta|^2$/n = $|\zeta|^2${2 ζ /n−1 + n−2}

Thus we have seen that a signed graph S(V,ζ) is complete if and only if

4$|\zeta|^2$ /n = $|\zeta|^2${2 ζ /n−1 + n−2}

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