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S-ITERATIVE SCHEME FOR ASYMPTOTICALLY NONEXPANSIVE MAPS IN BANACH SPACES

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ABSTRACT

In this article, we derive the necessary and sufficient conditions for the strong convergence of the iterative sequence generated by pair of non expansive and asymptotically non expansive mappings into their common fixed point.

Keywords: *Asymptotically Non expansive, Non expansive, Strong convergence.*

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I. INTRODUCTION AND PRELIMINARIES

Let E be a real Banach space and let C be a non-empty subset of E. Let T: $C \to C$ be a mapping.

Definition 1.1 The mapping T is said to be non expansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall \text{ x, y} \in \text{C}$$

Definition 1.2 The mapping T is said to be asymptotically non expansive if there is a sequence $\{k_n\} \subseteq [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \quad \forall \text{ x, y} \in \text{C} \text{ and } n \geq 0$$

In 1973,Petryshyn and Willianson [8] proved a necessary and sufficient condition for the strong convergence of the Mann iterative schemes to a fixed point of a quasi-non expansive mapping in a Hilbert space. Subsequently Liu [3, 4], extended the above results and obtained some necessary and sufficient conditions for an Ishikawa-type iterative scheme with errors to converge to a fixed point of an asymptotically quasi-non expansive map. Moore and Nnoli [5] proved necessary and sufficient conditions for the strong convergence of the Mann iteration process to a fixed point of an asymptotically demicontractive map in a real Banach space. Recently Zeng, Wong and Yao [13] proved necessary and sufficient conditions for the strong convergence of the Mann iteration process to a fixed point for pair of asymptotically demicontractive and quasi-non expansive mappings in a real Banach space. Their theorems thus improve and extend the results of Liu [3, 4], Osilike [6] and several others.

Theorem 1.3 [5] Let E be a real Banach space. Let T: E → E be a uniformly L-Lipschitzian asymptotically demicontractive map with a nonempty fixed point set $F(T)$. Suppose $\{a_n\}_{n\geq 0}$ is the sequence associated to the asymptotic demicontractivity of T and $\{c_n\} \subseteq [0,1]$ is a sequence such that $\sum_{n\geq 0} c_n^2 < \infty$ and $\sum_{n\geq 0} c_n (a_n^2 - 1) < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in$ E by $x_{n+1} = (1 - c_n)x_n + c_n T^n x_n$, n≥ 0 then $\{x_n\}$ converges strongly to a fixed point of T if and only if

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$\lim_{inf} d(x_n, F(T)) = 0$. In particular, $\{x_n\}$ converges strongly to an $x_0 \in F(T)$ if and only if there exists a subsequence of $\{x_n\}$ converging strongly to x_0.

Theorem 1.4 [13] Let C be a nonempty closed convex subset of a real Banach space E,

Let S: $C \to C$ be a quasi-non expansive mapping, and T: $C \to C$ be a L-Lipschitzian asymptotically S-demicontractive mapping with sequences $\{a_n\}$, $\{k_n\} \subseteq [1, \infty)$ and $\{\epsilon_n\} \subseteq [0, \infty)$. Suppose the common fixed point set F = F(S) $\cap$ F(T) $\neq \emptyset$, and there is a real sequence $\{c_n\} \subseteq [0,1]$ satisfying that $\sum_{n\geq 0} c_n^2 < \infty$, $\sum_{n\geq 0} c_n (a_n^2 - 1) < \infty$, $\sum_{n\geq 0} c_n (k_n - 1) < \infty$ and $\sum_{n\geq 0} c_n \epsilon_n < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in$ C by $x_{n+1} = (1 - c_n) S x_n + c_n T^n x_n$, n$\geq 0$. Then $\{x_n\}$ converges strongly to an element of F if and only if $\lim_{inf} d(x_n, F) = 0$. In particular, $\{x_n\}$ converges strongly to $x_0 \in F$ if and only if there exists an infinite subsequence of $\{x_n\}$ which converges strongly to $x_0 \in F$.

The following theorem is the main result in this paper, which extends and improves recent result of Zeng, Wong and Yao.

II. MAIN RESULT

Theorem 2.1 Let C be a non-empty closed convex subset of a real Banach space E. Let S :C $\to$ C be a non expansive mapping and let T: C $\to$ C be an asymptotically non expansive mapping with sequence $\{k_n\} \subseteq [1, \infty)$. Assume that F = F(S) $\cap$ F(T) $= \{x \in C / Sx = Tx = x\} \neq \emptyset$ and there is a sequence $\{c_n\} \subseteq [0,1]$ satisfying that $\sum c_n (k_n - 1) < \infty$. For arbitrary $x_0 \in$ C, let $\{x_n\}$ be a sequence iteratively defined by $x_{n+1} = S y_n$, $y_n = (1 - c_n) S x_n + c_n T^n x_n$, n $\geq$ 0. Then $\{x_n\}$ converges strongly to an element of F if and only if $\lim_{inf} d(x_n, F) = 0$. In particular, $\{x_n\}$ converges strongly to p $\in F$ if and only if there exists an infinite subsequence of $\{x_n\}$ which converges strongly to p $\in F$.

To prove our main result, we need the following lemma.

Lemma 2.2 (see [7]). Let $\{a_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq (1 + \gamma_n) a_n + \beta_n$, n$\geq$ 1 if $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n\geq 1}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Proof of the main result.

From definitions 1.1 and 1.2,

$$\begin{aligned} \|x_{n+1} - p\| &= \|S y_n - p\| \\ &\leq \|y_n - p\| \\ &\leq (1 - c_n)\|S x_n - p\| + c_n \|T^n x_n - p\| \\ &\leq (1 - c_n)\|x_n - p\| + k_n c_n \|x_n - p\| \\ &= (1 + \gamma_n)\|x_n - p\| \end{aligned}$$

Where $\gamma_n = c_n (k_n - 1)$. According to the condition that $\sum \gamma_n < \infty$, and also from lemma 2.2, we have $\lim_{n\to\infty}\|x_n - p\|$ exists for each p $\in F$, so that there exists K > 0 such that $\|x_n - p\| \leq K$ for all n $\geq$1, Consequently we obtain

$$d(x_{n+1}, F) \leq (1 + \gamma_n) d(x_n, F)$$

IJARSE
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It again follows from lemma 2.2 that $\lim_{n\to\infty} d(x_n, F)$ exists.

III. NECESSITY

If $\{x_n\}$ converges strongly to some point $p \in F$, then from $0 \le d(x_n, F) \le \|x_n - p\| \to 0$ as $n \to \infty$, we have $\lim_{inf} d(x_n, F) = 0$.

IV. SUFFICIENCY

If $\lim_{inf} d(x_n, F) = 0$, then from lemma 2.2 that $\lim_{n\to\infty} d(x_n, F) = 0$. Next we prove that $\{x_n\}$ is a cauchy sequence in C. Put $M = e^{\sum_{n\ge 0}\gamma_n}$ then $1 \le M < \infty$, since $\lim_{n\to\infty} d(x_n, F) = 0$ for given $\epsilon > 0$ there exists a positive integer N such that for all $n \ge N$, $d(x_n, F) < \frac{\epsilon}{2M}$.

In particular there exists $q \in F$ such that $\|x_N - q\| = d(x_N, q) < \frac{\epsilon}{2M}$.

Again, for all $n \ge N$

$$\begin{aligned}\|x_n - q\| &\le (1+\gamma_{n-1})\|x_{n-1} - q\| \\ &\le (1+\gamma_{n-1})(1+\gamma_{n-2})\|x_{n-2} - q\| \\ &\le \cdots \\ &\le \prod_{i=N}^{n-1}(1+\gamma_i)\ \|x_N - q\| \\ &\le e^{\sum_{i=N}^{n-1}\gamma_i}\|x_N - q\| \\ &\le M\|x_N - q\| \\ &< \frac{\epsilon}{2}\end{aligned}$$

Consequently, we deduce that for all $n \ge N, m \ge 1$

$$\|x_{n+m} - x_n\| \le \|x_{n+m} - q\| + \|x_n - q\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence in C, but C is closed we have $\lim_{n\to\infty} x_n$ exists due to the completeness of E. We may suppose that $\lim_{n\to\infty} x_n = x_0 \in C$. We now show that $x_0 \in F$. Indeed, for given $\epsilon_0 > 0$ there exists a positive integer $N_0 \ge N$ such that for all $n \ge N_0$

$d(x_n, F) < \frac{\epsilon_0}{2(1+k_1)}$ and $\|x_n - x_0\| < \frac{\epsilon_0}{2(1+k_1)}$.

Thus, there exists $y_0 \in F$ such that $\|x_{N_0} - y_0\| = d(x_{N_0}, y_0) < \frac{\epsilon_0}{2(1+k_1)}$

now we have,
$$\begin{aligned}\|Tx_0 - x_0\| &\le \|Tx_0 - Tx_{N_0}\| + \|Tx_{N_0} - y_0\| + \|y_0 - x_{N_0}\| + \|x_{N_0} - x_0\| \\ &\le (1+k_1)\|x_{N_0} - x_0\| + (1+k_1)\|x_{N_0} - y_0\| \\ &< (1+k_1)\frac{\epsilon_0}{2(1+k_1)} + (1+k_1)\frac{\epsilon_0}{2(1+k_1)} = \epsilon_0.\end{aligned}$$

And
$$\begin{aligned}\|Sx_0 - x_0\| &\le \|Sx_0 - Sx_{N_0}\| + \|Sx_{N_0} - y_0\| + \|y_0 - x_{N_0}\| + \|x_{N_0} - x_0\| \\ &\le 2\|x_{N_0} - x_0\| + 2\|x_{N_0} - y_0\| \\ &< 2\frac{\epsilon_0}{2(1+k_1)} + 2\frac{\epsilon_0}{2(1+k_1)} = \frac{2\epsilon_0}{2(1+k_1)}.\end{aligned}$$

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Since $\epsilon_0 > 0$ is arbitrary, we infer that $Tx_0 = x_0$ and $Sx_0 = x_0$. This completes the proof.

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