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MODELLING THE EFFECT OF DELAY IN A NONLINEAR INNOVATION DIFFUSION MODEL

Rakesh Kumar[1], Anuj Kumar Sharma[2], Kulbhushan Agnihotri[3]

[1,3]Department of Applied Sciences, S.B.S. State Technical Campus, Ferozepur-152004, Punjab (India)

[2]Department of Mathematics, L.R.D.A.V. College, Jagraon, Ludhiana-142026, Punjab (India)

ABSTRACT

In this paper, we formulate a mathematical model by a system of two delayed differential equations, explaining the rate of change of adopter and non-adopter population. The rate of change of adopters is affected by the variable external, variable internal influences, intrinsic growth rate of non-adopter population, their emigration rate or death rate etc. The model is analysed by using the stability theory and Hopf-bifurcation analysis by taking time delay as a bifurcation parameter. Further, the role of external influences in neutralizing the effect of evaluation period is studied. Some numerical simulations are carried out to support the theoretical findings.

Keywords: Innovation diffusion model, Delay, External influences, Stability analysis, Hopf-bifurcation.

I. INTRODUCTION

Diffusion is a special type of communication concerned with the spread of messages that are perceived as dealing with new ideas, and necessarily represent a certain degree of uncertainty to an individual or organization [1, 2, 3]. Models that rely on diffusion theory to predict the adoption of an innovation are called diffusion models. An innovation is an idea, practice, or object that is perceived as new by an individual or other unit of adoption. The modelling and forecasting of the diffusion of market innovations has been a topic of practical and academic interest since the 1960 [4, 5, 6]. Although the **Roger's model** of new product diffusion was widely accepted in the marketing literature yet it had several limitations. The limitations of **Roger's model** were first examined by Bass (1969) and he also proposed a model of diffusion in comparison to Rogers's approach.

The Bass Model is an extension of the other two and assumes that potential adopters (or adoption units) are influenced in their purchase behaviour by two sources of information: an external, like mass-media communication and an internal, word-of mouth. The most important parameters in the Bass Model are the market potential m, the coefficient p of external influence and the coefficient q of internal influence. Specifically, m is the total number of people who will eventually use the product, p is the likelihood that somebody who is not yet using the product will start using it because of mass media coverage or other external factors, q is the likelihood that somebody who is not yet using the product will start using it because of ''word-of-mouth'' or other influence from those already using the product. The model incorporating all the above mentioned factors developed by Bass is as follow:

$$\frac{dA(t)}{dt} = \left(p + q\,\frac{A(t)}{m} \right)(m - A(t))$$

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The Bass model has been increasingly used to understand the pattern of growth of adopters under the influence of mass media coverage or other external factors, word-of-mouth and other influences [2, 4, 6, 7, 8]. The awareness stage and a decision-making stage has also been incorporated into the Bass model [9, 10, 11, 12, 13]. To make Bass model more realistic, we will be generalizing it by incorporating evaluation period together with logistically growing non-adopter population. The model is formulated by taking into consideration two stages in place of five i.e., the evaluation stage and the adoption stage. It is supposed that shifting of non-adopter population to adopter population is not instantaneous rather it takes some evaluation period τ. The main aim of the paper is to see the impact of evaluation period on the diffusion innovation system.

II. THE MATHEMATICAL MODEL

Let the non -adopter and adopter population density at any time t be $x(t)$ and $y(t)$ respectively. Assume that τ is the average evaluation time for an individual to evaluate the product so as to decide whether to adopt it or not. Let d, μ, ϕ be the death rate of a population, the word of mouth of adopters of the product with potential consumers and the intensity of an advertisement of a product. Also, let v is the discontinuance rate of adopters of the product. The rate of change of adopters is just because of external as well as internal factors, their deaths, their rate of discontinuance to use the product. In this model, evaluation period τ is taken as a control parameter, to see the impact of evaluation period in understanding the pattern of the dynamics of the non-adopter and adopter population. Assume that the non-adopters population is logistically growing with intrinsic growth rate s and with carrying capacity L and also with survival probability through stage $e^{-d\tau}$. Thus, the governing equations for the model system are as follows:

$$\begin{aligned}\frac{dx}{dt} &= s\left(x-\frac{x^2}{L}\right)-(\phi+\mu y(t-\tau))x(t-\tau)e^{-d\tau}+vy(t)-dx(t)\\ \frac{dy}{dt} &= (\phi+\mu y(t-\tau))x(t-\tau)e^{-d\tau}-(d+v)y(t)\end{aligned} \qquad \text{.....(1)}$$

III. STABILITY OF STEADY STATES POINTS

There are three feasible steady state for the system, (i) $E_0\,(0,0)$ is the trivial steady state,(ii) $E_1(x,0)$ and (iii) $E_2(x^*,y^*)$.

At $E_0\,(0,0)$, the system is asymptotically stable provided $d > s$ and the condition is obvious.

For $E_1(x,0)$, where $x=\frac{L}{s}(s-d)$ exists if $s > d$ and sufficient conditions for the equilibrium point to be stable at $E_1(x,0)$ is $s > d$.

Also $E_2(x^*,y^*)$ is the positive steady state equilibrium point, where

$$x^* = \frac{(d+v)\,y^*}{(\phi+\mu y^*)e^{-d\tau}}$$

and y^* are the roots of the following equation :

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$$K_1 y^2 + K_2 y + K_3 = 0; \;..................(2)$$
$$where\;\; K_1 = L\mu^2 de^{-2d\tau}, K_2 = r(d+v)^2 + 2L\phi\mu e^{-2d\tau} - L\mu(d+v)(r-d)e^{-d\tau}$$
$$and\;\; K_3 = L\phi^2 de^{-2d\tau} - L\phi(d+v)(r-d)e^{-d\tau}$$

$$i.e., y^* = \frac{-K_2 \pm \sqrt{K_2{}^2 - 4K_1K_3}}{2K_1} \;\; is\;\; positive\;\; provided$$
$$s > d \;\; and \;\; d + v > Max\left\{\frac{\phi d}{s-d}, L\mu\right\}$$

IV. DYNAMICAL BEHAVIOUR

The characteristic equation of the variational matrix of the delayed innovation diffusion model system takes the form:

$$D(\lambda,\tau) = \left(\lambda^2 + \Delta_1\lambda + \Delta_2\right) + \left(\Delta_3\lambda + \Delta_4\right)e^{-\lambda\tau} = 0............(3)$$
$$where\;\; \Delta_1 = 2d + v - s\left(1 - \frac{2x}{L}\right);$$
$$\Delta_2 = d(d+v) - s\left(1 - \frac{2x}{L}\right)(d+v);$$
$$\Delta_3 = \mu + \phi(y - x);$$
$$\Delta_4 = s\left(1 - \frac{2x}{L}\right)(\mu + \phi x) + d\mu + d\phi(y - x);$$

For $\tau = 0$, the equation (3) becomes

$$\lambda^2 + (\Delta_1 + \Delta_3)\lambda + (\Delta_2 + \Delta_4) = 0 \;\; .... (4)$$

Therefore, all the roots of the characteristic eqn. will have negative real parts if

$$H_1 : \; \Delta_1 + \Delta_3 > 0 \;\;\; and \;\;\; H_2 : \;\; \Delta_2 + \Delta_4 > 0$$

Theorem1. The sufficient conditions for the local asymptotic stability without any evaluation period τ is that the Eqn. (4) shall have both roots negative is

$$d > s\left(1 - \frac{2x}{L}\right) \& \; \mu + \phi x > \phi y$$

For $\tau \neq 0$, we will study the dynamics of the system i.e., we want to determine if the real part of some root of Eqn.(3) increases to reach zero and eventually becomes positive as τ varies. This shows that the time delay τ i.e. evaluation period results in Hopf-bifurcation.

Theorem 2. [9] The necessary and sufficient conditions for $E_2(x^*, y^*)$ to be asymptotically stable in the presence of an evaluation period are

1. the real parts of all the roots of $D(\lambda, \tau) = 0$ are negative,
2. for all real ω and for $\tau > 0$, $D(\lambda, \tau) \neq 0$.

Proof. Assume that for some $\tau > 0$, $\lambda = i\omega$ ($\omega > 0$ and $i = \sqrt{-1}$) is a root of characteristic equation (3), where ω is a positive real number. If we substitute

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$\lambda = i\omega$ into (3), then we have

Real Part : $\Delta_4 \cos\omega\tau + \Delta_3 \omega \sin\omega\tau = \omega^2 - \Delta_2$

Imaginary Part : $\Delta_3 \omega \cos\omega\tau - \Delta_4 \sin\omega\tau = -\Delta_1 \omega$

Squaring and adding these eqns., we get

$$\omega^4 + \left(\Delta_1^2 - \Delta_3^2 - 2\Delta_2\right)\omega^2 + \left(\Delta_2^2 - \Delta_4^2\right) = 0 \quad \text{.............(5)}$$

If $H_3 :\ \left(\Delta_1^2 - \Delta_3^2 - 2\Delta_2\right) > 0 \ \ and \ \ \left(\Delta_2^2 - \Delta_4^2\right) > 0$,then the equation (5) will have negative roots. Therefore characteristic equation (4) will not have purely imaginary roots. Since H_1 and H_2 ensure that all roots of (4) have negative real parts. By Rouche's Theorem, it follows that all roots of (5) will have negative real parts too.

If $H_4 :\ \left(\Delta_2^2 - \Delta_4^2\right) < 0$, then from Routh-Hurwitz criterion, Eqn.(5) has a unique positive root ω_0^2. Under this condition, the characteristic equation (3) will have a pair of purely imaginary roots of the form $\pm i\omega_0$. Put ω_0^2 in real and imaginary parts and solving for τ, we get

$$\tau_n^* = \frac{1}{\omega_0}\cos^{-1}\frac{(\omega_0^2 - \Delta_2)\Delta_4 - \Delta_1\Delta_3\omega_0^2}{\Delta_3\omega_0^2 + \Delta_4^2} + \frac{2n\pi}{\omega_0};\ n = 0,1,2,3.... \quad \text{.........(6)}$$

Theorem3. [1, 10, 11] (a) If H_1−H_3 hold, then all roots of Eqn.(3) have negative real parts for $\tau \geq 0$.

(b) If H_1, H_2 and H_4 hold, then the equilibrium point $E_2\,(x^*, y^*)$ is asymptotically stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$ and as τ increases through τ_0, $E_2\,(x^*, y^*)$ bifurcates into small periodic solutions, where $\tau_0 = \tau_n^*$ for $n = 0$ is given by Eqn.(6).

For this purpose, let us now compute the transversality condition for Hopf-bifurcation, and we turn to showing that $\left\{\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right\}_{\tau=\tau_0^*,\omega=\omega_0} > 0$ and this will signify that there exists at least one eigen value with positive real part for $\tau > \tau_0^*$. Thus the conditions of producing Hopf-bifurcation are satisfied and yield the required periodic solution [6, 12, 13].

So, differentiate the transcendental Eqn. (3) w.r.t. τ, we have

$$sign\left\{\mathrm{Re}\frac{d\lambda}{d\tau}\right\}_{\tau=\tau_0^*,\lambda=i\omega_0} = sign\left\{\frac{\sqrt{\left(\Delta_1^2 - \Delta_3^2 - 2\Delta_2\right)^2 - 4\left(\Delta_2^2 - \Delta_4^2\right)}}{\left(\Delta_1^2\omega_0^2 + (\Delta_2 - \omega_0^2)^2\right)\left(\Delta_4^2 + \Delta_3^2\omega_0^2\right)}\right\}$$

Therefore, by virtue of condition H_4, we will have the transversality condition

$$\left\{\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right\}_{\tau=\tau_0^*,\omega=\omega_0} > 0\,.$$

V. NUMERICAL COMPUTATIONS

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In this section, we present numerical simulations of the system (1). We assume a hypothetical set of parametric values as s = 0.3258, v = 0.011, d = 0.11, ϕ = 0.15, μ = 0.22, L = 10 i.e.

$$\frac{dx}{dt} = 0.3258\left(x - \frac{x^2}{100}\right) - \left(0.15 + 0.22y(t-\tau)\right)x(t-\tau)e^{-0.11t} + 0.011y(t) - 0.11x(t)$$

$$\frac{dy}{dt} = \left(0.15 + 0.22y(t-\tau)\right)x(t-\tau)e^{-0.11t} - (0.11 + 0.011)x(t)$$

When the system is integrated with initial values (0.1, 0.1), the system converges to a stable equilibrium point E_2 *(0.2014, 0.3939)*, i.e., the system is locally asymptotically stable without any time delay (τ) and is shown in Fig. 1

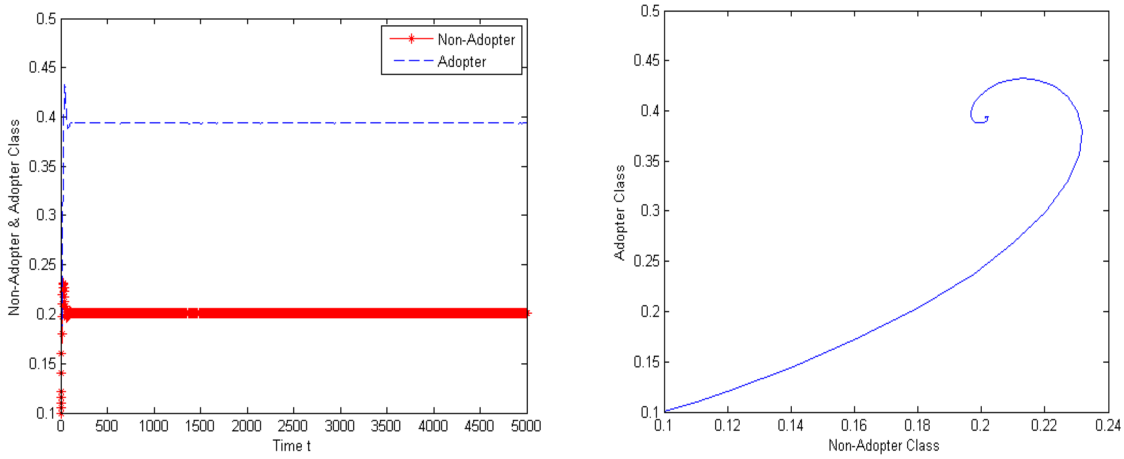

Fig.1 (Locally stable equilibrium for Non-Adopter and Adopter Class without any τ)

Numerically, by using the above said set of parametric values, for the delayed innovation diffusion system, a purely imaginary root $i\omega$ is calculated, and using it in (6), we will be able to find the critical value of evaluation period τ =1.0187 for the model system (1) such that E_2 *(0.2931, 0.5725)* loses its stability as τ passes through the critical value (taking other parameters fixed). Moreover at τ =2.1287, the transversality condition, for the existence of Hopf-bifurcation is also satisfied, which shows that the interior equilibrium E_2 remains stable for 0< τ < 2.1287 and becomes unstable for $\tau \geq$ 2.1287 . Fig 2 shows stable dynamics of the innovation diffusion system for τ =0.9871. An existence of Hopf-bifurcation in the form of limit cycle is shown for τ =2.1287 in Fig.3. Thus, it can be easily seen that there is a range of parameter for τ such that the system produces asymptotic stability around interior equilibrium E_2 for τ <2.1287 and as τ increases beyond this critical value of evaluation period, the system loses its stability and shows excitable nature in the form of limit cycle. Thus it indicates that the there is a threshold limit of evaluation period (τ, time taken by non-adopter population to become the member of the adopter population class) below which the system produces asymptotic stability and above it system shows excitability. Thus it can be concluded that the system around the interior equilibrium E_2 enters into a Hopf bifurcation and exhibits the cyclic nature for a certain amount of evaluation period. A more stable limit cycle of Non-Adopter and Adopter Class is shown for τ = 2.5287 in Fig.4.

Further, the effect of external influences to achieve maturity stage is also shown in Fig.5, i.e., when we make an increase in the cumulative density of external from p=0.15 to 0.25, the system converges to equilibrium point E_2(0.06241,0.1223) for Non-Adopter and Adopter populations.

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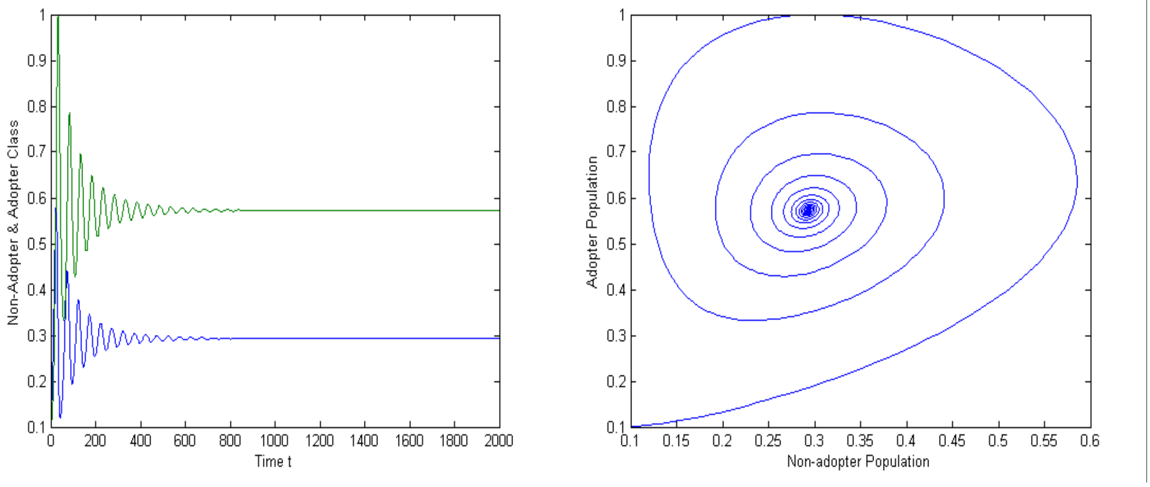

Fig.2 (Asymptotically stable equilibrium position for Non-Adopter and Adopter Classes at τ =1.81287)

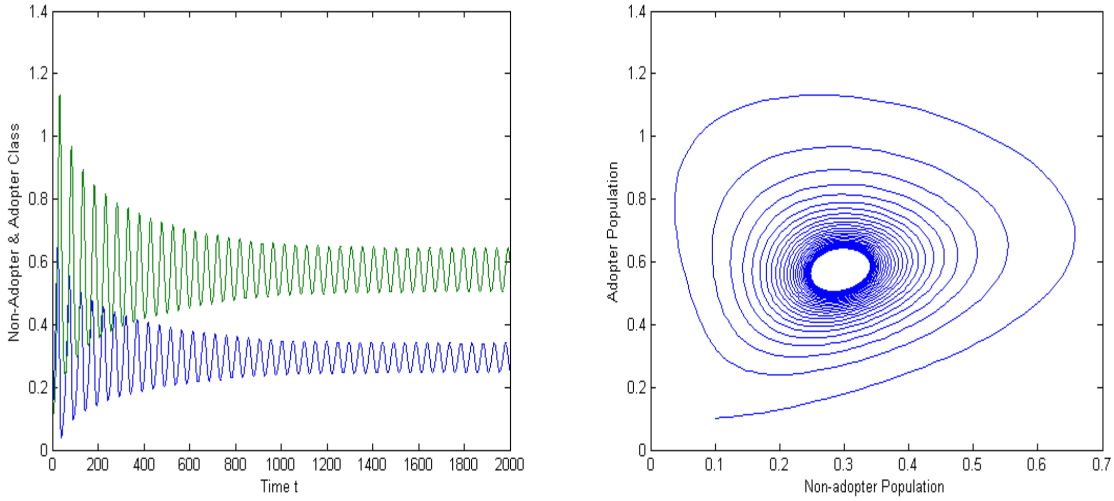

Fig.3 (Oscillatory character of Non-Adopter and Adopter Class is shown for critical value τ = 2.1287)

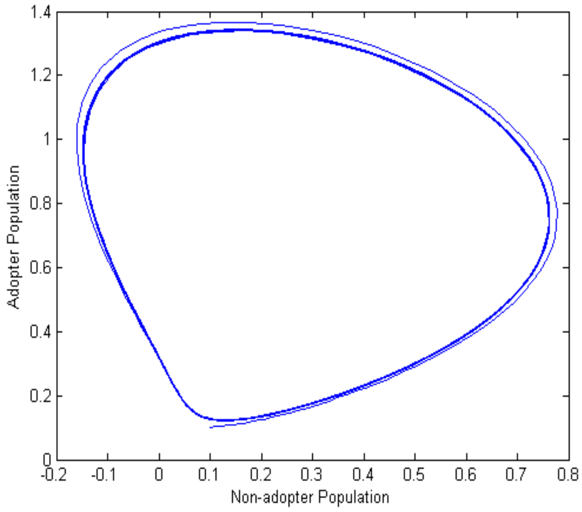

Fig.4 (A more stable limit cycle of Non-Adopter and Adopter Class is shown for τ = 2.5287)

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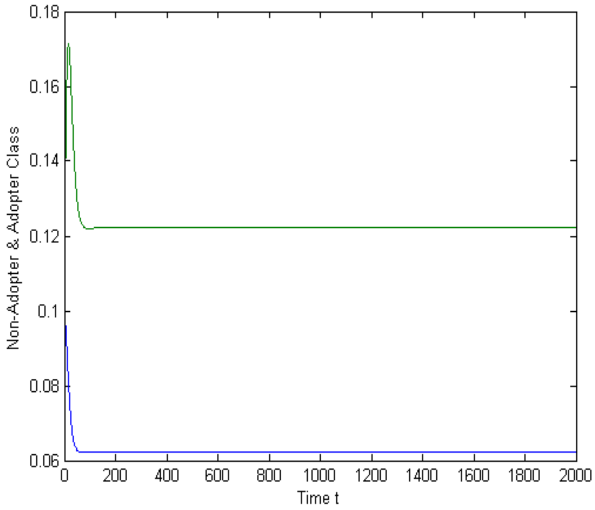

Fig.5 (Convergence of given system to E_2 (0.06241,0.1223) for Non-Adopter and Adopter populations at τ = 2.1287)

VI. CONCLUSION

In the present paper, our basic aim was to investigate the effect of evaluation period (time delay) on the innovation diffusion process of the system given by (1). In this paper, we have found that time delay has a vital role to play in establishing the periodic oscillations in the diffusion innovation system. It is observed that the system (1) was producing local asymptotic stability without the evaluation period (Fig.1) i.e., the given system does not have any excitable nature. Also the system is asymptotically stable for some evaluation period (Fig. 2). Further, we have been able to find threshold value of evaluation period, crossing over which Hopf-bifurcation is occurred, shown in Fig.3. Moreover, a more stable limit cycle of Non-Adopter and Adopter Class is shown for τ = 1.1287 in Fig. 4. It is clear that the non- adopters take average evaluation time to evaluate the product for adoption, while shifting over to the adopter class. The main effect of variable external influences is to make the equilibrium level of adopters population density reach to its equilibrium with a much faster rate (Fig.5).

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