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# FIXED POINT THEOREMS IN COMPACT FUZZY METRIC SPACE

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## ABSTRACT

In this Paper we introduce a set termed $E_\alpha$ and studied its properties in fuzzy metric spaces for using this concept we have proved the results on fixed point in compact fuzzy metric spaces for two self mappings without using the iteration method. Pal and Pal [6], Rathore, Dolas and Singh [7] and Rathore, Singh, Rathore and Singh [8] in these papers we pick up this idea.

## I INTRODUCTION

For any positive number $\alpha$ and a self mapping of a metric space X with metric $\rho$, Kannan [3, 4 ] furnished a set termed

$$S_\alpha = S_\alpha \{ z \in X, \rho(z, Tz) \leq \alpha \}$$

and proved certain properties on it and also established the well knownBanach's Fixed Point Theorem (Liusternik and Sobole [5] ) and certain of its extensions (Edelstein[1]) admit of alternative proofs. In 1993, Pal and Pal [6] introduced a set $E_\alpha$ defined as follows

$E_\alpha = \{x \in X$ , for each x , $\exists$ $y \in X$ such that $\rho(x, Ty) + \rho(y, Tx) \leq \alpha\}$ for a self mapping in a different way. They studied with meticulous care and established some results on fixed points without iteration method.

In the same manner, recently Rathore et.al [7] expand the concept of $E_\alpha = \{x \in X$ , for each x , $\exists$ $y \in X$ such that $\rho(x, Ty) + \rho(y, Sx) \leq \alpha\}$

for two self mappings and studied it properties for the existence of fixed points, without iteration method. With all these fruitful investigation in the background we naturally got interested in generalizing the concept of $E_\alpha$in fuzzy metric space and proved its properties and other related propositions yielding fixed points.

## PRELIMINARIES

**Definition 1.[2]** Let (X, M, *) be a fuzzy metric space. We define **open ball B(x, r, t)** with centre $x \in X$ and radius r, $0 < r < 1$, $t > 0$ as

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$$B(x, r, t) = \{ y \in X ; M(x, y, t) > 1 - r \}.$$

**Definition 2[2].** Let (X, M, *) be a fuzzy metric space. We define **closed ball B [x, r, t]** with centre $x \in X$ and radius r, $0 < r < 1$, $t > 0$ as

$$B[x, r, t] = \{y \in X ; M(x, y, t) \geq 1 - r \}.$$

**Topology induced by fuzzy metric**

Let (X, M, *) be a fuzzy metric space. Define

$\tau = \{ A \subset X : x \in A$ if and only if there exist $t > 0$ and r,

$0 < r < 1$ such that $B(x, r, t) \subset A\}$

Then $\tau$ is a topology on X.

**Result 3. [2].** Every open ball is an open set and every closed ball is a closed set.

**Definition 4.** Let (X, M, ∗) be fuzzy metric space. Then a collection

$C = \{G_\alpha : \alpha \in \Lambda\}$ of subsets of X is said to be a **cover** of X if

$$\bigcup_{\alpha \in \Lambda} G_\alpha = X.$$

**Definition 5**. Let (X, M, ∗) be a fuzzy metric space. A covering C of X is said to be a **open covering** of X if every member of C is an open set.

**Definition6**. A fuzzy metric space (X, M, ∗) is said to be **compact** if every open covering of X has a finite subcovering.

Now, we prove some theorems.

**Theorem 7.** A closed subset of a compact fuzzy metric space is compact.

**Proof.** Let (X, M, ∗) be a compact fuzzy metric space and Y be a non-empty closed subset of X. We shall show that Y is compact.

Let $C = \{G_\alpha : \alpha \in \Lambda\}$ be an open covering of Y in X.

Since Y is closed then $Y^C$ is open in X therefore

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$$X = Y \bigcup Y^C$$

$$= \left( \bigcup_{\alpha \in \Lambda} G_\alpha \right) \bigcup Y^C$$

$\Rightarrow \{G_\alpha : \alpha \in \Lambda\} \bigcup Y^C$ is an open covering of X.

$\Rightarrow \exists$ a finite subcovering $\{ G_{\alpha_i} : \alpha_i \in \Lambda, i = 1, 2, \ldots\ldots.n \} \bigcup Y^C$ of X.

Since $Y \subset X$ and $Y^C$ covers no part of Y, $Y \subset \bigcup_{i=1}^{n} G_{\alpha_i}$.

Therefore $\{ G_{\alpha_i} : \alpha_i \in \Lambda, i = 1, 2, \ldots\ldots, n \}$ is a finite sub covering of Y.

It follows that Y is compact.

**Results 8 [2]** In a fuzzy metric space every compact set is closed and bounded.

**Theorem 9.** If $\{K_n\}$ is a sequence of compact set in a fuzzy metric space (X, M, *) such that $K_n \supset K_{n+1}$ (n = 1, 2, 3, …) and if $\lim_{n \to \infty}$ diameter $(K_n) = 1$

then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

**Proof:** Suppose on the contrary that $\bigcap_{n=1}^{\infty} K_n = \phi$, then

$$\left( \bigcap_{n=1}^{\infty} K_n \right)^c = (\phi)^c$$

$$\Rightarrow \left( \bigcup_{n=1}^{\infty} K_n^c \right) = X$$ [By De-morgan's Law]

$\Rightarrow \left\{ K_n^c \right\}$ is an open covering of X

[ $K_n^s$ are closed in X, because of being compact in X ]

$\Rightarrow \left\{ K_n^c \right\}$ is also an open cover of $K_1$

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$\Rightarrow K_1 \subset K^c_{n_1} \cup \ldots\ldots \cup K^c_{n_k}$ [since $K_1$ is compact]

$\Rightarrow K_1 = \left(K_1 \cap K^c_{n_1}\right) \cup \ldots\ldots\ldots\ldots \cup \left(K_1 \cap K^c_{n_k}\right)$

$\Rightarrow K^c_1 = \left[\left(K_1 \cap K^c_{n_1}\right) \cup \ldots\ldots\ldots\ldots \cup \left(K_1 \cap K^c_{n_k}\right)\right]^c$

$\Rightarrow \phi = \left(K^c_1 \cup K_{n_1}\right) \cap \ldots\ldots\ldots\ldots \cap \left(K^c_1 \cup K_{n_k}\right)$

[complement are taken w.r.t. $K_1$]

$\Rightarrow \phi = K^c_1 \cup \left(K_{n_1} \cap \ldots\ldots\ldots \cap K_{n_k}\right)$

$\Rightarrow K^c_1 = \phi$ and $K_{n_1} \cap \ldots\ldots\ldots \ldots \cap K_{n_k} = \phi$,

which is a contradiction because $K_n \supset K_{n+1}$ .

Now, for the completeness of the proof it is sufficient to show that $\bigcap_{n=1}^{\infty} K_\alpha$ contains no other point except x.

Suppose, on the contrary that $\exists\, y \in X$

such that $y \neq x$ and $y \in \bigcap_{n=1}^{\infty} K_n$ .

Then M(y, x, t) < 1, we write $\delta = M(y, x, t)$.

Since $\lim_{n \to \infty} \operatorname{diam} K_n = 1$

$\Rightarrow \exists\ q \in N$ such that diam $K_q > 1 - \delta$

$\Rightarrow$ diam $K_q > \delta$. (1)

Again x, y $\in K_q$ , we have

M(x, y, t) $\geq \operatorname{diam} K_q$ (2)

From (1) and (2) and by the definition of $\delta$

M(x, y, t) > M(x, y t)

which is a contradiction.

Hence $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

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Now, we introduce the following

**Definition10.** Let (X, M, *) be a Fuzzy metric space and $\alpha$ be a positive number. Also let T and S be operator mappings X into itself. Then we define $E_\alpha$ to be the set of all those point of X, for each point x of which there exists a point $y \in X$ such that

$$M(x, Ty, t) * M(y, Sx, t) \geq 1- \alpha.$$

It may be noted that $y \in E_\alpha$

**PROPERTIES OF THE SET $E_\alpha$**

**Proposition 11 :** Let X be fuzzy metric space and T and S be continuous mappings of X into itself. Then $E_\alpha$ is closed set.

**Proof.** If $E_\alpha$ is empty then $E_\alpha$ is closed. Now if $E_\alpha$ is non-empty then let $\{y_n\}$ be a sequence of points of the set $E_\alpha$ converging to $z \in X$. In order to prove that $E_\alpha$ is closed, we shall show that $z \in E_\alpha$.

Since $\{y_n\}_{n=1}^{\infty} \to z$

$\Rightarrow$ for any $\varepsilon > 0$, $\exists$ a positive $m \in N$ such that

$$M(y_n, z, t) \geq 1- \varepsilon , \qquad \forall\, n \geq m . \tag{1}$$

Since T and S is continuous then

$$M(Ty_n, Tz, t) \geq 1- \varepsilon \tag{2}$$

$$M(Sy_n, Sz, t) \geq 1- \varepsilon \tag{3}$$

and since $y \in E_\alpha$, then there exist $y \in X$ such that

$$M(y_n, Ty, t) * M(y, Sy_n, t) \geq 1- \alpha_n , \quad \text{for each } n \in N.$$

Now,

$$M(z, Ty_n, t) * M(y_n, Sz, t) \geq M(z, y_n, t/_3) * M(y_n, Tz, t/_3) * M(Tz, Ty_n, t/_3) *$$

$$M(y_n, z, t/_3) * M(z, Sy_n, t/_3) * M(Sy_n, Sz, t/_3)$$

$$> (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon) * (1 - \alpha_n)$$

since $\varepsilon$ is arbitrary. This implies that

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$M(z, Ty_n, t) * M(y_n, Sz, t) \geq 1- \alpha_n$ .

Therefore $z \in E_\alpha$ .

It follows that $E_\alpha$ is closed.

**Proposition 12.** Let T and S be a continuous mappings of the compact fuzzy metric space (X, M, *) into itself. Let $\{\alpha_n\}$ be a decreasing sequence of positive number converging to zero. Then the set $E_{\alpha_n}$ (n = 1, 2, ……) is non-void if and only if there exist z, y $\in$ X such that x = Ty and y = Sx.

**Proof. Necessary condition**

Let x and y be points in X such that x = Ty and y = Sx, then

$$M(x, Ty, t) * M(y, Sx, t) = 1 * 1$$

$$= 1 \geq 1- \alpha_n \ (n = 1, 2, \ldots\ldots)$$

This implies that $x \in E_{\alpha_n}$ (n = 1, 2, 3, ….) and consequently $E_{\alpha_n}$ is non-void.

**Sufficient Condition**

Let $\{\alpha_n\}$ be a decreasing sequence of positive numbers converging to zero and $E_{\alpha_n}$ (n=1, 2, 3, ….) be non-void.

Suppose $x_n \in E_{\alpha_n}$ (n = 1, 2, 3, …..); $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ converging to $x \in X$. For each $x_{n_i}$ (i = 1, 2, 3, …….) there exist a point $y_{n_i} \in X$ such that

$$M\left(x_{n_i}, Ty_{n_i}, t\right) * M\left(y_{n_i}, Sx_{n_i}, t\right) \geq 1 - \alpha_{n_i}.$$

We have a subsequence $\{y_{n_{i_r}}\}$ of $\{y_{n_i}\}$ such that $\{y_{n_{i_r}}\}$ is convergent to $y \in X$. Therefore, for $\varepsilon > 0$

$$M(x, Ty, t) * M(y, Sx, t) \geq M\left(x, x_{n_{i_r}}, \frac{t}{2}\right) * M\left(x_{n_{i_r}}, Ty, \frac{t}{2}\right)$$

$$M\left(y, y_{n_{i_r}}, \frac{t}{2}\right) * M\left(y_{n_{i_r}}, Sx, \frac{t}{2}\right)$$

$$\geq M\left(x, x_{n_{i_r}}, \frac{t}{2}\right) * M\left(x_{n_{i_r}}, y_{n_{i_r}}, \frac{t}{6}\right)$$

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$$M\left(Ty_{n_{i_r}}, Ty, \frac{t}{6}\right) * M\left(y, y_{n_{i_r}}, \frac{t}{2}\right)$$

$$M\left(y_{n_{i_r}}, Sx_{n_{i_r}}, \frac{t}{6}\right) * M\left(Sx_{n_{i_r}}, Sx, \frac{t}{6}\right)$$

Taking limit as n $\to\infty$ and using the continuity of *, we have

$$M(x, Ty, t) * M(y, Sx, t) \geq 1 * M\left(x_{n_{i_r}}, Ty_{n_{i_r}}, \frac{t}{6}\right) * 1 * 1$$

$$M\left(y_{n_{i_r}}, Sx_{n_{i_r}}, \frac{t}{6}\right) * 1$$

$$\geq M\left(x_{n_{i_r}}, Ty_{n_{i_r}}, \frac{t}{6}\right)\left(y_{n_{i_r}}, Sx_{n_{i_r}}, \frac{t}{6}\right)$$

$$\geq 1 - \alpha_{n_{i_r}}. \qquad (1)$$

Letting n $\to\infty$ in (1), we have

$$M(x, Ty, t) * M(y, Sx, t) = 1.$$

Hence M(x, Ty, t) = 1 and M(y, Sx, t) = 1 leading to the conclusion

$$Ty = x \text{ and } y = Sx$$

This completes the proof.

**Proposition 13.** Let T and S be two continuous mapings of the compact fuzzy metric space (X, M, *) into itself. Also let $\{\alpha_n\}$be a decreasing sequence of positive numbers converging to zero and let the sequence $\{E_{\alpha_n}\}$ of sets be such that diam $\{E_{\alpha_n}\}\to 1$ as n $\to\infty$. A necessary and sufficient condition for the existence of a common fixed point of T and S in X is that the sets $E_{\alpha_n}$ (n = 1, 2, …) are non-void.

**Proof.** First we suppose that T and S have a common fixed point in X.

Let $x \in X$ be such that Tx = x = Sx. Now

$$M(x, Ty, t) * M(y, Sx, t) = 1 * 1$$

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$$= 1 > 1 - \alpha_n \ (n = 1, 2, \ldots).$$

This implies $x \in E_{\alpha_n}$ . Hence $E_{\alpha_n}$ are non-void. In virtue of its property $E_{\alpha_n}$ is compact. Also

$E_{\alpha_{n+1}} \subset E_{\alpha_n}$ and $\operatorname{diam}\{E_{\alpha_n}\} \to 1$ as $n \to \infty$.

So, $\bigcap_{n=1}^{\infty} E_{\alpha_n}$ contains exactly one point.

Let $x_0 \in \bigcap_{n=1}^{\infty} E_{\alpha_n}$ . Then there exists $x_n \in E_{\alpha_n}$ (n = 1, 2, ….) such that

$$M(x_0, Tx_n, t) * M(x_n, Sx_0, t) \geq 1 - \alpha_n \quad (1)$$

Since X is compact, then sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$ (say). Let $x_{n_i} \to x \in X$ as $i \to \infty$

Thus by (1)

$$M(x_0, Tx_{n_i}, t) * M(x_{n_i}, Sx_0, t) \geq 1 - \alpha_n . \quad (2)$$ Letting $i \to \infty$, we

have

$$M(x_0, Tx, t) * M(x, Sx_0, t) = 1 \geq 1 - \alpha_n . \quad (3)$$ So, $x \in E_{\alpha_n}$ (n = 1, 2, 3,

…) i.e. $x \in \bigcap_{n=1}^{\infty} E_{\alpha_n}$ .

Since $\bigcap_{n=1}^{\infty} E_{\alpha_n}$ contains exactly one point, we obtain $x = x_0$.

Therefore (3) implies that

$$M(x_0, Tx_0, t) = 1 \text{ and } M(x, Sx_0, t) = 1$$

and hence $Tx_0 = x_0$ and $Sx_0 = x_0$

Thus $x_0$ is a common fixed point of Tand S in X.

.**Proposition12.** Let (X, M, *) be a compact fuzzy metric space and let T and S be two continuous mappings of X into itself. Let $\{\alpha_n\}$ be a decreasing sequence of position numbers converging to zero. Let X contain no points x, y with the property that x = Ty and y = Sx.Then the sets $E_{\alpha_n}$ is empty for sufficiently large values of n.

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**Proof.** Suppose the theorem is not true. then there exist exit positive integers $n_1, n_2, \ldots$ with $n_1 < n_2 < n_3, \ldots\ldots$ such that none of the set $E_{n_k}$ (K = 1, 2, …..) is empty. Let $x_{n_k} \in E_{n_k}$, k = 1, 2, …..Since X is compact, there exists a subsequence $\{x_{n_{k_i}}\}$ of the sequence $\{x_{n_k}\}$ such that $\{x_{n_{k_i}}\}$ is convergent in X. Let $\{x_{n_{k_i}}\}$ converge to x in X.

Then for each $x_{n_{k_i}}$ (i = 1, 2, …..) there exists $y_{n_{k_i}} \in X$ such that

$$M(x_{n_{k_i}}, Ty_{n_{k_i}}, t) * M(y_{n_{k_i}}, Sx_{n_{k_i}}, t) \geq 1 - \alpha_{n_{k_i}}.$$

Since X is compact, there exists a subsequence $\{y_{n_{k_{i_r}}}\}$ of $\{y_{n_{k_i}}\}$. Such that $\{y_{n_{k_{i_r}}}\}$ is convergent. Let $y_{n_{k_{i_r}}} \to y$ in X. Now

$$M\left(x_{n_{k_{i_r}}}, Ty_{n_{k_{i_r}}}, t\right) * M\left(y_{n_{k_{i_r}}}, Sx_{n_{k_{i_r}}}, t\right) \geq 1 - \alpha_{n_{k_{i_r}}}$$

Letting $r \to \infty$

$$M(x, Ty, t) * M(y, Sx, t) \geq 1.$$

So, $M(x, Ty, t) \geq 1$ and $M(y, Sx, t) \geq 1$

$\Rightarrow M(x, Ty, t) = 1$ and $M(y, Sx, t) = 1$.

Consequently $Ty = x$ and $Sx = y$ .

This leads to a contradiction.

Hence the set $E_{\alpha_n}$ are empty for sufficiently large value of n.

**Main result**

Pal and Pal [6] has given the following theorem:

**Theorem 1.** Let $(X, \rho)$ be a compact metric space. Let T be a continuous map of X into itself. Suppose that

$$\rho(Tx,Ty) \leq \beta[\rho(x,Ty) + \rho(y,Tx)],$$

for every $x, y \in X$ and for a real number $\beta$ with $0 < \beta < 1/2$.

Then there exists exactly one point $x_0 \in X$, such that $T(x_0) = x_0$.

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Now, we are proving the following theorems taking the clues from above theorem for compact fuzzy metric space.

**Theorem 2.** Let (X, M, *) be a compact fuzzy metric space and T, S be two continuous self mappings on X. Suppose that $t * a * a \geq a$.

$$M(Sx, Ty, t) \geq M\left(x, Ty, \frac{t}{k}\right) * M\left(y, Sx, \frac{t}{k}\right), \text{ for every } x, y \in X, t > 0.$$

Then there exists exactly one point $x_o \in X$ such that $T(x_o) = x_o$ and $S(x_o) = x_o$.

**Proof**. Let $\{\alpha_n\}$ be a decreasing sequence of positive numbers converging to zero. In view of its property $E_{\alpha_n}$ (n = 1, 2, 3, ….) is compact. Clearly

$$E_{\alpha_{n+1}} \subset E_{\alpha_n};\ n = 1, 2, 3, \ldots\ldots$$

Now, we shall show that diam ( $E_{\alpha_n}$ ) $\rightarrow 1$ as $n \rightarrow \infty$

For any $x, y \in E_{\alpha_n}$, $k \in (0, 1)$ and $\forall\, t > 0$

$$M(x, y, t) \geq M\left(x, Sx, \frac{t}{3}\right) * M\left(Sx, Ty, \frac{t}{3}\right) * M\left(Ty, y, \frac{t}{3}\right)$$

$$\geq M\left(x, Sx, \frac{t}{3}\right) * M\left(x, Ty, \frac{t}{3k}\right) * M\left(y, Sx, \frac{t}{3k}\right) * M\left(Ty, y, \frac{t}{3}\right)$$

$$\geq M\left(x, Sx, \frac{t}{3}\right) * M\left(x, y, \frac{t}{6k}\right) * M\left(y, Ty, \frac{t}{6k}\right) * M\left(y, x, \frac{t}{6k}\right) * M\left(x, Sx, \frac{t}{6k}\right) * M\left(Ty, y, \frac{t}{3}\right)$$

$$\Rightarrow M(x, y, t) \geq M\left(x, Sx, \frac{t}{6k}\right) * M\left(y, Ty, \frac{t}{6k}\right) * M\left(x, y, \frac{t}{6k}\right)$$

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By repeated applications of above inequality m-time, we get

$$M(x, y, t) \geq M\left(x, Sx, \frac{t}{6k}\right) * M\left(y, Ty, \frac{t}{6k}\right) * M\left(x, y, \frac{t}{6k^m}\right)$$

Since $M\left(x, y, \frac{t}{6k^m}\right) \to 1$ as $m \to \infty$ it follow that

$$M(x, y, t) \geq M\left(x, Sx, \frac{t}{6k}\right) * M\left(y, Ty, \frac{t}{6k}\right)$$

Now, for $x \in E_{\alpha_n}$ , then there exists a point $z \in X$ such that

$$M(x, Tz, t) * M(z, Sx, t) \geq 1- \alpha_n.$$

And for $y \geq E_{\alpha_n}$ , then there exists a point $z' \in X$ such that

$$M(y, Tz', t) * M(z', Sy, t) \geq 1- \alpha_n .$$

So,

$$M(x, y, t) \geq M\left(x, Tz, \frac{t}{12k}\right) * M\left(Tz, Sx, \frac{t}{12k}\right) *$$

$$M\left(y, Sz', \frac{t}{12k}\right) * M\left(Sz', Ty, \frac{t}{12k}\right)$$

$$\geq M\left(x, Tz, \frac{t}{12k}\right) * M\left(x, Tz, \frac{t}{12k}\right) *$$

$$M\left(z, Sx, \frac{t}{12k}\right) * M\left(y, Sz', \frac{t}{12k}\right) *$$

$$M\left(z', Ty, \frac{t}{12k}\right) * M\left(y, Sz', \frac{t}{12k}\right)$$

$$\geq \left[M\left(x, Tz, \frac{t}{12k}\right) * M\left(z, Sx, \frac{t}{12k}\right)\right] *$$

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$$\left[M\left(z', Ty, \frac{t}{12k}\right) * M\left(y, Sz', \frac{t}{12k}\right)\right]$$

$$\geq (1 - \alpha_n) * (1 - \alpha_n) \qquad [\text{since } z' \in E_{\alpha_n}]$$

Therefore $M(x, y, t) \geq 1 - \alpha_n$ .

Since $\{\alpha_n\}$ is a sequence converging to zero as $n \to \infty$, then $M(x, y, t) \to 1$

Hence diam $(E_{\alpha_n}) \to 1$ as $n \to \infty$.

Thus $\{E_{\alpha_n}\}$ is a sequence of sets, such that

(i). $E_{\alpha_n}$ is perfect

(ii). $E_{\alpha_{n+1}} \subset E_{\alpha_n}$

(iii). diam $(E_{\alpha_n}) \to 1$ as $n \to \infty$

So, by the theorem [9]

$\bigcap_{n=1}^{\infty} E_{\alpha_n}$ contains exactly one point.

Let $x_0 \subset \bigcap_{n=1}^{\infty} E_{\alpha_n}$

Now, we shall show that $x_0$ is a fixed point in X.

Since $x_0 \in E_{\alpha_n}$ (n = 1, 2, 3, …..), then there exists a point $x_n \in X$ such that

$$M(x_0, Tx_n, t) * M(x_n, Sx_0, t) \geq 1 - \alpha_n \forall\, t > 0 \qquad (1)$$

Let $\{x_{n_i}\}$ be a subsequence of the sequence $\{x_n\}$ such that the sequence $\{x_{n_i}\}$is convergent let the

$\{x_{n_i}\}$ convergen to x in X. Then

$$M(x, x_0, kt) \geq M\left(x, x_{n_i}, \frac{t}{2}\right) * M\left(x_{n_i}, x_0, \frac{t}{2}\right)$$

$$\geq M\left(x, x_{n_i}, \frac{t}{2}\right) * M\left(x_{n_i}, Sx_0, \frac{t}{6}\right) *$$

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$$M\left(Sx_0, Tx_{n_i}, \frac{t}{6}\right) * M\left(Tx_{n_i}, x_0, \frac{t}{6}\right)$$

$$\geq M\left(x, x_{n_i}, \frac{t}{2}\right) * M\left(x_{n_i}, Sx_0, \frac{t}{6}\right) * M\left(x_0, Tx_{n_i}, \frac{t}{6k}\right) *$$

$$M\left(x_{n_i}, Sx_0, \frac{t}{6k}\right) * M\left(x_{n_i}, Sx_0, \frac{t}{6}\right)$$

$$\geq M\left(x, x_{n_i}, \frac{t}{2}\right) * \left[M\left(x_{n_i}, Sx_0, \frac{t}{6k}\right) * M\left(x_0, x_{n_i}, \frac{t}{6k}\right)\right]$$

$$\geq M\left(x, x_{n_i}, \frac{t}{2}\right) * (1 - \alpha_{n_i}).$$

Taking $i \to \infty$

$M(x, x_0, t) \geq 1 * 1$

$M(x, x_0, t) \geq 1$

$\Rightarrow M(x, x_0, t) = 1$

Hence $x = x_0$

Now from condition (1)

$M(x_0, Tx_{n_i}, t) * M(x_{n_i}, Sx_0, t) \geq 1 - \alpha_n$ .

On letting $i \to \infty$

$M(x_0, Tx, t) * M(x, Sx, t) = 1$

$\Rightarrow \quad M(x_0, Tx_0, t) * M(x_0, Sx_0, t) = 1$

this implies $M(x_0, Tx_0, t) = 1$ and $M(x_0, Sx_0, t) = 1$

i.e. $Tx_0 = x_0 = Sx_0$

**Uniqueness**

Let $y_0$ be another common fixed point of S and T, then

$M(x_0, y_0, t) = M(Sx_0, Ty_0, t)$

$$\geq M\left(x_0, Ty_0, \frac{t}{k}\right) * M\left(y_0, Sx_0, \frac{t}{k}\right)$$

$$\geq M\left(x_0, y_0, \frac{t}{k}\right) * M\left(y_0, x_0, \frac{t}{k}\right)$$

$M(x_0, y_0, kt) = M(x_0, y_0, t)$.

This implies that $x_0 = y_0$.

This completes the proof.

## REFRENCE


[1] **Edelstein, M.** : On fixed and periodic points under contractive mappings, *J. London Math. Soc.*, 39 (1962), 74-79.

[2] **George, A.** and **Veeramani, P.** : On same result in Fuzzy metric space, *Fuzzy sets and system,* 64 (1994), 395-399.

[3] **Kannan, R.** : On certin sets and fixed point theorems, *Rev. roun. Pures Et* Appl. XIV, 1,51, Bucarest (1969).

[4] **Kannan, R.** :Some results on fixed point, *Amer. Math. Monthly*, 76, 405 (1969).

[5] **Liusternik, L.A.** and **Sobole, V.J**. :*Elements of Functional, New York,* 27 (1961).

[6] **Pal, M.** and **Pal, M.C.** : On certain sets and fixed point theorems, *Bull.Cal. Math. Soc.*, 85 (1993), 301-310.

[7] **Rathore, M.S., Dolas, U.** and **Singh, B.** :Propertiesofcertain sets Eα and fixed point theorems in compact metric spaces, *Vikram MathematicalJournal* 18 (1998), 72-83.

[8] **Rathore, M.S., Singh, M., Rathore, S.** and **Singh, N.** : Concept of the set Eα and common fixed points, *Bull. Cal. math. Soc.*, 94(4) (2002), 259-270.

[9] **Rathore, M.S., Singh, M., Rathore, S.** and **Singh, N.** : Concept of the set Eα and common fixed points, *Bull. Cal. math. Soc.*, 94(4) (2002), 259-270.