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FIXED POINTS OF EXPANSIVE MAPPINGS IN QUASI PARTIAL METRIC SPACES

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ABSTRACT

In this paper, we study some xed point theorems in quasi partial metric spaces using expansive mappings. Also, we derive some common xed point theorems for two compatible mappings in this framework. The results improve and generalize many results existing in the literature. Some examples and an application to solve a rst order ordinary di erential equation have also been presented to illustrate the e ectiveness of obtained results.

***Keywords*: Fixed Point, Quasi Partial Metric Space, Expansive Mappings.**

I INTRODUCTION AND PRELIMINARIES

In 1984, Wang et al. [1] introduced expansive mappings and established some xed point theorems for complete metric spaces. His result can be stated as:

Theorem 1.1 [1] Let $T : X \to X$ be an onto mapping defined on a complete metric space (X, d) satisfying the condition

$$d(T a, T b) \geq c\, d(a, b) \text{ for all } a, b \in X.$$

where $c > 1$. Then T has a unique fixed point in X.

Later on, various authors including Khan et al. [2], Rhoades [3], Kang [4] etc. extended this result in various ways.

In 1994, Matthews [5] introduced partial metric spaces with an application in denotational semantics and program verification. Till Now, there exists so many generalizations for partial metric spaces. For details, see [6], [7], [8], [9], [10].

Karapinar generalized this notion by presenting Quasi partial metric spaces in [11]. Let us recall that for a nonempty set X, a mapping $q : X \times X \to R^+$ is said to be a Quasi partial metric if the following conditions hold:

(q1) $0 \leq q(x, x) = q(x, y) = q(y, y)$, then $x = y$;

(q2) $q(x, x) \leq q(x, y)$;

(q3) $q(x, x) \leq q(y,x)$;

(q4) $q(x, z) \leq q(x, y) + q(y, z) - q(y;,y)$

for all $x, y \in X$. Then the pair (X, q) is called a Quasi partial metric space.

If $q(y, x) = q(x, y)$ for each $x, y \in X$, then (X, q) reduces to partial metric space. Also, for a quasi-partial metric q on X, the mapping $d_q : X\times X \to R_+$ defined by

IJARSE
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$$d_q(x,y) = q(x, y) + q(y, x) - q(x, x) - q(y, y)$$

is called a (usual) metric on X.

Karapinar et al. [11] introduced the following definitions and results in his work.

Definition 1.2 [11] Let (X, q) be a quasi partial metric space. Then

(1) a sequence $\{x_n\}$ in X converges to x (in X) iff

$$q(x, x) = \lim_{n\to\infty} q(x, x_n) = \lim_{n\to\infty} q(x_n, x);$$

(2) a sequence $\{x_n\}$ in X is called a Cauchy sequence iff $\lim_{m,n\to\infty} q(x_m, x_n)$ and $\lim_{n,m\to\infty} q(x_n, x_m)$ exist and are finite;

(3) the quasi partial metric space (X, q) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that

$$q(x,x) = \lim_{m,n\to\infty} q(x_m, x_n) = \lim_{n,m\to\infty} q(x_n, x_m).$$

Lemma 1.3 [11] Let (X, q) be a QPMS. Let (X, p_q) be the corresponding PMS and let (X, d_{pq}) be the corresponding metric space. The following statements are equivalent:

(1)The sequence $\{x_n\}$ is cauchy in (X, q) .

(2)The sequence $\{x_n\}$ is cauchy in (X, p_q).

(3)The sequence $\{x_n\}$ is cauchy in (X, d_{pq}).

Lemma 1.4 [11] Let (X, q) be a QPMS. Let (X, p_q) be the corresponding PMS and let (X, d_{pq}) be the corresponding metric space. The following statements are equivalent:

(1) (X, q) is complete.

(2) (X, p_q) is complete.

(3) (X, d_{pq}) is complete.

Moreover,

$$\lim_{n\to\infty} d_{pq}(x,x_n) = 0 \leftrightarrow p_q(x, x) = \lim_{n\to\infty} p_q(x, x_n) = \lim_{n,m\to\infty} p_q(x_n, x_m)$$
$$\leftrightarrow q(x,x) = \lim_{n\to\infty} q(x, x_n) = \lim_{n,m\to\infty} q(x_n, x_m)$$
$$= \lim_{n\to\infty} q(x_n, x) = \lim_{m,n\to\infty} q(x_m, x_n).$$

Lemma 1.5 [11] Let (X, q) be a quasi partial metric space. Then q(x, y) = 0 implies x = y and if $x \neq y$, then q(x, y) > 0 and q(y, x) > 0.

The main purpose of this paper is to introduce the notion of expansive mappings in Quasi partial metric spaces and to establish some fixed point theorems in this setup. Also, some comparative examples and an application to solve first order ordinary differential equations are also given to illustrate the usability of obtained results.

II QUASI PARTIAL METRIC AND EXPANSIVE MAPPINGS

The following lemma will be helpful in proving our main result.

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Lemma 2.1 Let (X, q) be a quasi partial metric space and $\{x_n\}$ be a sequence of points of X. If there exists a number $k \in (0,1)$ such that

$$q(x_{n+1}, x_n) \leq kq(x_n, x_{n-1}) \; ; n = 1, 2,\ldots \qquad (2.1)$$

Then $\{x_n\}$ is a cauchy sequence in X. **Proof**. By given condition (2.1), we obtain

$q(x_{n+1}, x_n) \leq kq(x_n, x_{n-1}) \leq k^2 q(x_{n-1}, x_{n-2}) \leq \ldots \leq k^n q(x_1, x_0)$.

Also, $\max \{q(x_n, x_n), q(x_{n+1}, x_{n+1})\} \leq q(x_{n+1}, x_n) \leq k^n q(x_1, x_0)$.

Then

$$\begin{aligned} d_q(x_n, x_{n+1}) &= q(x_n, x_{n+1}) + q(x_{n+1}, x_n) - q(x_n, x_n) - q(x_{n+1}, x_{n+1}) \\ &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_n) + q(x_n, x_n) + q(x_{n+1}, x_{n+1}) \\ &\leq k^n q(x_0, x_1) + k^n q(x_1, x_0) + k^n q(x_1, x_0) + k^n q(x_1, x_0) \\ &= 3k^n q(x_1, x_0) + k^n q(x_0, x_1) \text{ where } k < 1 \end{aligned}$$

$$\Rightarrow \lim n \to \infty \; d_q(x_n, x_{n+1}) = 0.$$

Similarly, we can show that

$$\lim n \to \infty \; d_q(x_{n+1}, x_n) = 0.$$

Further,

$$\begin{aligned} d_q(x_n, x_m) &= d_q(x_n, x_{n+1}) + d_q(x_{n+1}, x_{n+2}) + \ldots + d_q(x_{m-1}, x_m) \\ &\leq 3k^n q(x_1, x_0) + k^n q(x_0, x_1) + 3k^{n+1} q(x_1, x_0) + k^{n+1} q(x_0, x_1) \\ &\quad + \ldots + 3k^{m-1} q(x_1, x_0) + k^{m-1} q(x_0, x_1) \\ &= 3k^n q(x_1, x_0)[1 + k + \ldots + k^{m-1}] + k^n q(x_0, x_1)[1 + k + \ldots + k^{m-1}] \\ &\leq (3k^n /1-k) q(x_1, x_0) + (k^n /1-k) q(x_0, x_1) \end{aligned}$$

This shows that $\{x_n\}$ is a cauchy sequence in X w.r.t. metric d_q. From Lemma 1.3, $\{x_n\}$ is cauchy in quasi partial metric space (X, q).

Theorem 2.2. Let (X, q) be a complete quasi partial metric space and $D : X \to X$ be a bijective mapping defined on X. Suppose that there exists $c_1, c_2, c_3 \geq 0$ such that $c_1 + c_2 + c_3 > 1$

and

$$q(Dx, Dy) \geq c_1 q(x, y) + c_2 q(x, Dx) + c_3 q(y, Dy) \text{ for all } x, y \text{ in } X. \qquad (2.2)$$

Then D has a fixed point in X.

Proof. Let $x_0 \in X$. Since D is bijective, there exists $x_1 \in X$ such that $Dx_1 = x_0$. Define a sequence $\{x_n\}$ in X such that $x_{n-1} = Dx_n$; n = 1,2,… If $x_{n-1} = x_n$ for some n, then the result is trivial. Therefore, assume that $x_{n-} \neq x_n$ for all n.

By given condition,

$$\begin{aligned} q(x_n, x_{n-1}) &= q(Dx_{n+1}, Dx_n) \\ &\geq c_1 q(x_{n+1}, x_n) + c_2 q(x_{n+1}, Dx_{n+1}) + c_3 q(x_n, Dx_n) \\ &= c_1 q(x_{n+1}, x_n) + c_2 q(x_{n+1}, x_n) + c_3 q(x_n, x_{n-1}) \end{aligned}$$

$$\Rightarrow (1 - c_3) q(x_n, x_{n-1}) \geq (c_1 + c_2) q(x_{n+1}, x_n)$$

$$\Rightarrow q(x_{n+1}, x_n) \leq ((1 - c_3) /(c_1 + c_2)) q(x_n, x_{n-1})$$

IJARSE
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Since c1 + c2 $\neq$ 0 and (1 - c3) > 0, therefore

$$q(x_{n+1}, x_n) \leq \lambda q(x_n, x_{n-1}) \text{ where } \lambda = ((1-c_3)/(c_1+c_2)) < 1.$$

Thus, by above lemma, $\{x_n\}$ is a cauchy sequence in X and since (X, q) is complete, therefore, (X, d_q) is complete where d_q is the usual metric induced by quasi metric q.

Therefore, $\{x_n\}$ is convergent in X w.r.t. metric dq. Let x ϵ X be such that

$$\lim n\to\infty\ d_q(x^*, x_n) = 0.$$

By Lemma 1.4, we have

$$q(x^*, x^*) = \lim n\to\infty\ q(x^*, x_n) = \lim n,m\to\infty\ q(x_n, x_m) \quad (2.3)$$

$$= \lim n\to\infty\ q(x_n, x^*) = \lim m,n\to\infty\ q(x_m, x_n).$$

Again by Lemma 2.2, the sequence $\{x_n\}$ is Cauchy in (X,d_q) i.e.

$$\lim m,n\to\infty\ d_q(x_m, x_n).$$

Also,

$$\max\{q(x_n, x_n), q(x_{n+1}, x_{n+1})\} \leq q(x_{n+1}, x_n)$$

$$\leq \lambda\, q(x_n, x_{n-1})$$

$$\leq \ldots \leq \lambda^n q(x_1, x_0).$$

Therefore, $\lim n\to\infty\ q(x_n, x_n) = 0$.

By definition of metric dq, we have, $\lim m, n\to\infty\ q(x_m, x_n) = 0$ and $\lim n,m\to\infty\ q(x_n, x_m) = 0$. Thus, by (2.3), we obtain

$q(x^*, x^*) = , \lim n\to\infty\ q(x^*, x_n) = , \lim n\to\infty\ q(x_n, x^*) = 0.$

which shows that $\{x_n\}$ is convergent in quasi partial metric space (X, q). Let u ϵ X be such that $x^* = Du$. Next we shall prove that $x^* = u$. Let us suppose that $x^* \neq u$. hen

$$q(x_n, x^*) = q(Dx_{n+1}, Du)$$

$$\geq c_1 q(x_{n+1}, u) + c_2 q(x_{n+1}, Dx_{n+1}) + c_3 q(u, Du).$$

As $n\to\infty$, above inequality becomes

$0 = q(x^*, x^*) \geq c_1 q(x^*, u) + c_3 q(u, x^*)$

$\Rightarrow c_1 q(x^*, u) + c_3 q(u, x^*) \leq 0.$

but as $x^* \neq u$, we have, $q(x^*, u) > 0$ and $q(u, x^*) > 0$ with c_1, c_3 non negative.

Thus, we arrive at a contradiction. Therefore, $x^* = u$ and hence

$$x^* = Du = u.$$

This completes the proof.

IJARSE
ISSN (O) 2319 - 8354
ISSN (P) 2319 - 8346

Corollary 2.3 Let (X, q) be a complete quasi partial metric space and $D : X \to X$ be a bijective mapping. Suppose that there exists a constant $c > 1$ such that

$$q(Dx, Dy) \geq c\, q(x; y) \text{ for all } x, y \text{ in } X.$$

Then D has a unique fixed point in X.

Proof. From above theorem, it follows that D has a fixed point x in X by putting $c_1 = c$ and c_2, $c_3 = 0$ in inequality (2.2).

For uniqueness, let z be another fixed point of D. Then,

$$q(x^*, z) = q(Dx^*, Dz) \geq c\, q(x^*, z) \text{ where } c > 1.$$

which is a contradiction. Therefore, $x^* = z$.

Corollary 2.4 Let (X, q) be a quasi partial metric space with q as complete metric and $D : X \to X$ be a bijective mapping de ned on X. Suppose that there exists a positive integer n and a constant $c > 1$ such that

$$q(D^n x, D^n y) \geq c\, q(x, y) \text{ for all } x, y \text{ in } X.$$

Then D has a unique fixed point in X.

Proof. From Corollary 2.3, D^n has a unique fixed point x^* in X. Also, $D^n(D x^*) = D(D^n x^*) = D x^*$ which shows that $D x^*$ is also a fixed point of D^n. By uniqueness of fixed point in D^n, we have $D x^* = x^*$ and thus D^n and D both have a unique fixed point x^*.

Example 2.5 Let $X = R^+$ and $q(x, y) = \max\{x- y, y- x\} + x$. Then (X, q) is a complete quasi partial metric space. Let $Dx = 3x^2$ for all x in X.

Note that D is a bijective mapping. Also, for all $x \leq y$, we obtain

$$q(Dx, Dy) = q(3x^2, 3y^2) = 3y^2$$
$$\geq c\, q(x, y) \text{ where } c = 3 > 1.$$

Thus, the condition of expansion is also satisfied for D.

Hence, all the conditions of Corollary 2.3 are fulfilled. Therefore, there exists a unique fixed point of D. Here, 0 is the unique fixed point.

III APPLICATION

Consider the periodic boundary value problem

$$\begin{cases} u'(t) = k(t, u(t)), & t \in I = [0, T] \\ u(0) = u(T) \end{cases} \qquad (3.1)$$

where $k : I \times R \to R$ is a continuous function and $T > 0$.

A function $u \in C^1(I, R)$ satisfying the above conditions is a solution to problem (3.1). This application presents the existence of a unique solution for given problem with some suitable conditions.

Theorem 3.1. Consider the periodic boundary value problem given in (3.1) where $k : I \times R \to R$ is a continuous function. Let us suppose that there exists $\lambda > 1$ such that $k(t, v) + \lambda v - k(t, u) - \lambda u \geq \lambda(v - u) + u$ for all u, v $\in R$ with $v \geq u$.

Then, existence of a solution of given problem ensures the existence of a unique solution for the same.

Proof. The given problem (3.1) can be rewritten as

IJARSE
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$$\begin{cases} u'(t) = \lambda u(t) = k(t, u(t)), \ t\epsilon\, I = [0, T] \\ u(0) = u(T) \end{cases}$$

which is equivalent to the integral equation

$$u(t) = \int_0^t G(t,s)[k(s,u(s)) + \lambda u(s)]ds$$

Where G(t,s)= $\{\frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, \ 0 \le s < t \le T$ and $\frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, \ 0 \le t < s \le T.$

Define a bijective mapping F(u(t)) $=\lambda \int_0^T \int_0^t G(t,s)[k(s,u(s)) + \lambda u(s)]ds, t \in I, u \in C(I,R)$.

If $u \in C(I,R)$ is a fixed point of F, then $u \in C(I,R)$ is a solution of the given problem.

Let us assume that $C(I,R) = U$. Define the metric q on U as

q(u, v) = $sup_{t\in I}|u(t) - v(t)| + v(t), u, v \in U, t \in I.$

Note that (U, q) is a complete quasi partial metric space. Using definition of G, we obtain for u ≥ v,

$$q(F(u), F(v)) = sup_{t\in I}|F(u(t)) - F(v(t))| + F(v(t))$$

$$\ge sup_{t,s\in I}\left|\begin{matrix} \lambda\int_0^T G(t,s)[k(s,u(s)) + \lambda u(s)]ds \\ -\lambda\int_0^T G(t,s)[k(s,v(s)) + \lambda v(s)]ds \end{matrix}\right|$$

$$= sup_{t,s\in I}\left|\lambda\int_0^T G(t,s)[k(s,u(s)) + \lambda u(s) - k(s,v(s)) - \lambda v(s)ds]ds\right|$$

$$\ge sup_{t,s\in I}|\lambda(u(s) - v(s)) + v(s)]ds|$$

Since $G(t,s) > 0, \lambda > 1$ and $u(s) \ge v(s)$, therefore the above inequality becomes

$$q(F(u), F(v)) \ge \lambda\int_0^T sup_{t,s\in I}G(t,s).\ \lambda[\{sup_{t,s\in I}|u(s) - v(s)|\} + v(s)]ds$$

$$= \lambda\int_0^T sup_{t,s\in I}G(t,s).\ \lambda\, q(u,v)ds$$

$$= \lambda^2 q(u,v)\int_0^T sup_{t,s\in I}G(t,s)\ ds$$

$$=\lambda^2 q(u,v) sup_{t,s\in I}\frac{1}{e^{\lambda t}-1}\left(\frac{1}{\lambda}\left|e^{\lambda(T+s-t)}\right|_0^t + \frac{1}{\lambda}\left|e^{\lambda(s-t)}\right|_t^T\right)$$

$$=\lambda^2 q(u,v) \times \frac{1}{\lambda} = \lambda q(u,v)$$

$\Rightarrow$ q(F(u), F(v)) ≥ $\lambda q(u,v)$.

This implies that F is an expansive mapping.

Thus all the required conditions for Corollary 2:3 are fulfilled. Consequently, F has a unique fixed point and thus given problem (3.1) possesses a solution.

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