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HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF P-VALENT FUNCTION ASSOCIATED WITH RUSCHWEYH DERIVATIVE OPERATOR

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ABSTRACT

The object of this paper is to use Toeplitz deter-minant to obtain a sharp upper bound of the second Hankel determinant $|a_{p+1}a_{p+3} - a^2_{p+2}|$ *for the p-valent functions be- longing to the class* S (λ , *β)*

1. Introduction, Definition and Motivation

Let A_p (p is a fixed integer ≥ 1) denote the class of analytic functions f(z) of the form,

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{1.1}$$

defined on the open unit disk:

$$\mathbb{U} = \{Z \in \mathbb{C} : |z| < 1\} \tag{1.2}$$

and let $A_1 = A$. Let S be the subclass of A consisting of univalent function $\mathbb{U}$. A function f(z) $\in A_p$ is said to be p-valent starlike function $(\frac{f(z)}{z} \neq 0)$ if it satisfies the condition ,

$$\Re\left\{\frac{zf'(z)}{pf(z)}\right\} > 0 \quad (\mathbb{Z} \in \mathbb{U}) \tag{1.3}$$

Key words and phrases: P-Valent Function, Upper Bound, Hankel Determinant, Toeplitz Determinant.

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The set of all these function is denoted by S_p^*. If is observed that for P = 1 then S_p^* reduces to S^*. For

$$\begin{aligned} f_z &= \sum_{m=0}^{\infty} a_m z^m \\ g_z &= \sum_{m=0}^{\infty} b_m z^m \end{aligned} \tag{1.4}$$

are analytic in $\mathbb{U}$ then the Hadamard product or convolution f(z) $*$ g(z) of f(z) and g(z) is defined by,

$$f(z) * g(z) = \sum_{h=0}^{\infty} a_m b_m z^m \tag{1.5}$$

Let,

$$\begin{aligned} D^{\lambda+p-1} f(z) &= \frac{z^p}{(1-z)^{p+\lambda}} * f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma\lambda + k}{\Gamma\lambda + p(k-p)!} a_k z^k \quad \lambda > -p, \mathbb{Z} \in \mathbb{U} \end{aligned} \tag{1.6}$$

This symbol $D^{\lambda+p-1}f(z)$ is called the Ruscheweyh derivative of f(z) and was introduced by Ruscheweyh [4].

The q^{th} determinant for $q \geq 1$ and $n \geq 1$ is stated by Noonan and Thomas [23] as,

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & & & \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.7}$$

This determinant has also been considered by several authors. For example, Noorin [24] determinant the rate of growth of $H_q(n)$ as $n \to \infty$ for function f (1.1) with bounded boundary. Ebrenbary in [6] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were deicussed by Layman's article[9]. It is well known that [5] for

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f $\in$ S and given by (1.2) the sharp inequality $|a_3 a_2^2| \leq 1$ holds. This corresponds to the hankel determinant with q = 2 and k = 1.

After that, Fekete-Szego further generalized the estimate $|a_3 - \mu a_2^2|$ with real μ and f $\in$ S for a given class of functions in A the sharp bound for the non linear function $|a_2 a_4 - a_3^2|$ is known as the Second Hankel Determinant.

Second Hankel determinant for various subclass of analytic functions were obtained by different researchers including Janteng et al. [12], Mishra and Gochhyay [21] and Murugusundaramoorthy and Mangesh [22] for some more works are [2][3][4].

Recently Trailokya Panigrahi consider the Hankel determinant in the case q = z, n = p+1 denoted by H_2(p+1) given by,

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2 \qquad (1.8)$$

Motivated by the above mentioned results obtained by different researchers in this direction. In this paper we obtain as sharp upper bounds to functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for function f belonging to certain subclass of p-valent functions defined as follows,

Definition 1.1. *A function f(z) $\in A_p$ is said to be in the class $S_p(\lambda, \beta)$ if it satisfies the condition,*

$$\Re\left\{(1-\beta)\left(\frac{D^{\lambda+p-1}f(z)}{z^p}\right) + \beta\left(\frac{(D^{\lambda+p-1}f(z))'}{pz^{p-1}}\right)\right\} > 0 \qquad (1.9)$$
$$(\beta \leq 1, \quad \mathbb{Z} \in \mathbb{U}, \quad \lambda > -p)$$

Note that, for p = 1, $\lambda = 0$, reduce the class,

$$(1-\beta)\frac{f(z)}{z} + \beta f'(z) > 0 \qquad (1.10)$$

it is class R(β) studied by Murugusundramurthi and Magesh [22].

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2. Preliminary Lemmas

Let P denote the class of function of the form,

$$P(z) = 1 + c_1 z + c_2 z^2 + \dots$$
$$= 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.1}$$

which are regular in $\mathbb{U}$ and satisfy $\mathrm{R}(\mathrm{p(z)}) > 0$ for any $\mathbb{Z} \in \mathbb{U}$. Here p(z) is called Caratheodory function [8]. To prove our main result, we need the following lemma,

Lemma 2.1. *[5] If $p \in P$ then,*

$$|C_k| \leq 2 \quad for \quad each \quad k \in N \tag{2.2}$$

Lemma 2.2. *[18][19] Let $p \in P$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$
$$4c_3 = \left\{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2 z)\right\}$$

For some values of x,z such that $|x| \leq 1$ and $|z| \leq 1$.

3. Main Results

Theorem 3.1. *If $f(z) \in S_p(\lambda, \beta)$ ($\beta \leq 1$, $\lambda >$ -p) then,*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{16p^2}{(p + 2\beta)^2(\lambda + p)^2(\lambda + p + 1)^2} \tag{3.1}$$

Proof. Let f(z) given by (1) be the class of $\mathrm{S}_p(\lambda, \beta)$. Then there exist an analytic function $\mathrm{p} \in \mathrm{P}$ in the unit disk $\mathbb{U}$ with p(0) = 1 and $\mathrm{R}\{p(z)\} > 0$ such that,

$$(1 - \beta)\left[\frac{D^{\lambda+p-1}f(z)}{z^p}\right] + \beta\left[\frac{(D^{\lambda+p-1}f(z))'}{pz^{p-1}}\right] = p(z) \tag{3.2}$$

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$$
\begin{aligned}
&[1-\beta]\left[1+\sum_{k=p+1}^{\infty}\frac{\Gamma\lambda+k}{\Gamma\lambda+p(k-p)!}a_k z^{k-p}\right]\\
&+\beta\left[1+\sum_{k=p+1}^{\infty}\frac{\Gamma\lambda+k}{\Gamma\lambda+p(k-p)!}\frac{k}{p}a_k z^{k-p}\right]\\
&=1+c_1z+c_2z^2+c_3z^3+\ldots
\end{aligned}
\tag{3.3}
$$

After simplification equating the coefficient of the like powers z, z^2, z^3, ... respectively on both sides we get,

$$a_{p+1}=\frac{pc_1}{(p+\beta)(\lambda+p)} \tag{3.4}$$

$$a_{p+2}=\frac{2pc_2}{(p+2\beta)(\lambda+p)(\lambda+p+1)} \tag{3.5}$$

$$a_{p+3}=\frac{6pc_3}{(p+3\beta)(\lambda+p)(\lambda+p+1)(\lambda+p+2)} \tag{3.6}$$

It is easily established that,

$$
\begin{aligned}
|a_{p+1}a_{p+3}-a_{p+2}^2| = &\left[\frac{6p^2c_1c_3}{(p+\beta)(p+3\beta)(\lambda+p)^2(\lambda+p+2)}\right.\\
&\left.-\frac{4p^2c_2^2}{(p+2\beta)^2(\lambda+p)^2(\lambda+p+1)^2}\right]
\end{aligned}
\tag{3.7}
$$

Applying Lemma, substituting for c_2 and c_3 and since $|c1| \leq 2$ by Lemma 2.1, let $c_1 = c$ assume without restriction $\in$ [0,2]we obtain,

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$$|a_{p+1}a_{p+3}-a_{p+2}^2| = \left|\frac{6p^2c\left[\frac{c^3+2c(4-c^2)x-c(4-c^2)x^2+2(4-c^2)(1-|x|^2z)}{4}\right]}{(p+2\beta)^2(\lambda+p)^2(\lambda+p+1)^2} - \frac{4p^2\left[(\frac{c^2+x(4-c^2)}{2})^2\right]}{(p+2\beta)^2(\lambda+p)^2(\lambda+p+1)^2)}\right|$$

$$\leq \left\{\frac{6p^2c^4+12c(4-c^2)x-6c(4-c)^2x^2+12(4-c^2)(1-|x|^2z)}{4(p+\beta)(p+3\beta)(\lambda+p)^2(\lambda+p+1)(\lambda+p+2)} + \frac{p^2[c^4+2c^2x(4-c^2)+x^2(4-c^2)^2]}{(p+2\beta)^2(\lambda+p)^2(\lambda+p+1)^2}\right\} \tag{3.8}$$

$$|x|=\rho \quad |z|\leq 1 \tag{3.9}$$

$$\leq \left\{\frac{6p^2c^4+12c(4-c^2)\rho-6c(4-c^2)^2\rho^2+12(4-c^2)(1-\rho^2)}{4(p+\beta)(p+3\beta)(\lambda+p)^2(\lambda+p+1)(\lambda+p+2)} + \frac{p^2[c^4+2c^2\rho(4-c^2)+\rho^2(4-c^2)^2]}{(p+2\beta)^2(\lambda+p)^2(\lambda+p+1)^2}\right\} = F(\rho) \tag{3.10}$$

$$F(\rho) = \frac{6p^2c^4+12c(4-c^2)\rho-6\rho^2(4-c^2)(c-2)+12(4-c^2)}{4(p+\beta)(p+3\beta)(\lambda+p)^2(\lambda+p+1)(\lambda+p+2)} + \frac{p^2[c^4+2c^2\rho(4-c^2)+\rho^2(4-c^2)^2]}{(p+2\beta)^2(\lambda+p)^2(\lambda+p+1)^2} \tag{3.11}$$

$$F'(\rho) = \frac{12(4-c^2)-2\rho\times 6(4-c^2)(c-2)}{4(p+\beta)(p+3\beta)(\lambda+p)^2(\lambda+p+1)(\lambda+p+2)} + \frac{p^2[2c^2(4-c^2)+2\rho(4-c^2)^2)]}{(p+2\beta)^2(\lambda+p)^2(\lambda+p+1)^2} \tag{3.12}$$

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with elementary calculus we can show that, $F'(\rho) > 0$ for $\rho > 0 \Rightarrow$ that F is an increasing function and thus for upper bound for (3.11) corresponds to $\rho = 1$, $c = 0$ gives,

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{16p^2}{(p+2\beta)^2(\lambda+p)^2(\lambda+p+1)^2} \tag{3.13}$$

□

Corollary 3.1. *For $\lambda = 0$, and $p = 1$,*

$$|a_2a_4 - a_3^2| \leq \frac{4}{(1+2\beta)^2} \tag{3.14}$$

result obtain by G. Murugusundaramoorthy et al.[22].

Corollary 3.2. *If $\lambda = 0$, $p = 1$, $\beta = 1$, then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{9} \tag{3.15}$$

the sharp result obtain by A. Janteng [13].

Conclusion. *In this paper we have obtained the sharp upper bounds for the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for functionbelonging to$S_p(\lambda, \beta)$. Further we are working on to find the upper bounds for convex function and also find the third Hankel determinant.*

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