International Journal of Advance Research in Science and Engineering
Vol. No. 5, Issue No. 08 , August 2016
www.ijarse.com
IJARSE
ISSN 2319 - 8354

COMMON FIXED POINT THEOREMS SATISFYINGAN IMPLICIT RELATION

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ABSTRACT

This paper is composed of two sections to study a systematic, comparative and comprehensive description of space and compact fuzzy metric space. Compact fuzzy 2-metric space deals with certain types of implicit relations.

An extensive study in fuzzy metric space and fuzzy 2-metric space had been done by Sharma [1] using this concept we have accomplished the task of generalizing the result of Aliouche [2] in the context of compact fuzzy metric space compact fuzzy 2-Metric spaces.

***Keywords**: Class of Implicit Relation, Compact Fuzzy Metric Spaces, Compact Fuzzy 2-Metric Space. weak-compatible.*

I. INTRODUCTION

The theory of fuzzy set, ows its origin to the classical paper of L. A. Zadeh [3]. There is a large community of mathematicians who have expansively developed the theory of fuzzy sets and their applications, especially Kaleva and Seikkala [4] and Kramosil and Michalek [5] have introduced the concept of fuzzy metric space in different ways. Recently, Sharma [1], Jain [6] have made efforts to elaborate the results of fuzzy metric space into fuzzy 2-metric . Using this concept we have accomplished the task of generalizing the result of Aliouche [2] in the context of compact fuzzy metric space, compact fuzzy 2-metric space.

For systematically, comparative and comprehensive study, we have divided our study in two sections.Section first is devoted to the results concerning with fixed point theorem related with implicit relation in compact fuzzy metric space.

Gahlar [7] who furnished the concept of 2-metric space, and this concept has prospered very fast in various directions. It is to be remarked that Sharma, Sharma and Iseki [8] studied for first time contraction type mapping in 2-metric space. Wenzhi [9] and many others preceded the study of probabilistic 2-metric space. We know that a 2-metric is a real valued function with domain $X \times X \times X$, whose abstract properties were suggested by the area function in Euclidean space. The method of introducing this is naturally different from 2- metric space theory; here we have to use simplex theory from algebraic topology. Sections second is based on comprehensive study of compact fuzzy 2-metric space with the help of a fixed point theorems related with implicit relation.

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A CLASS OF IMPLICIT RELATION

Let F_6 bethe family of all continuous mapping $F(t_1, t_2, t_3, t_4, t_5, t_6) : R^6_+ \to R$

satisfying the following conditions:

(C_1) : $\forall$ $u \geq 0$, $v > 0$ and $w \geq 0$ with

(C_a) : $F(u, v, v, u, w, 1) > 0$ or

(C_b) : $F(u, v, u, v, 1, w) > 0$, we have $u > v$.

(C_2) : For all $u < 1$, $F(u, 1, 1, u, u, 1) < 0$.

(C_3) : For all $u < 1$, $F(u, u, 1, 1, u, u) < 0$.

MAIN RESULTS

1.1 Common Fixed Point Theorems In Compact Fuzzy Metric Space

Theorems 1.1 A, B, S and T be self mappings of a compact fuzzy metric space $(X, M, *)$ satisfying:

(a) $S(X) \subset B(X)$ and $T(X) \subset A(X)$,

(b) the pair (S, A) and (T, B) are weak-compatible,

(c) S and A are continuous,

(d) $F[M(Sx, Ty, t), M(Ax, By, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(Sx, By, t)] > 0$

$\forall$ $x, y \in X$ and $F \in F_6$ satisfies (C_1) , (C_2) and (C_3) for which one of $M(Ax, By, t)$, $M(Ax, Sx, t)$ and $M(By, Ty, t)$ is positive. Then A, B, S and T have a unique common fixed point in X.

Proof. Let $K = \text{Sup}\{ M(Ax, Sx, t) \ ; \ x \in X, t > 0\}$.
Since X is compact fuzzy metric space, then there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that

$$\lim_{n\to\infty} M(Ax_n, Sx_n, t) = K, \quad \forall\ t > 0.$$

Since,
$M(Ax_0, Sx_0, 3t) \geq M(Ax_0, Ax_n, t) * M(Ax_n, Sx_n, t) * M(Sx_n, Sx_0, t)$

by the continuity of A and S and $\lim_{n\to\infty} x_n = x_0$

we get,

$$M(Ax_0, Sx_0, 3t) \geq K, \quad \forall\ t > 0.$$

Hence, $M(Ax_0, Sx_0, t) = K$. [by the definition of K]

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Since $S(X) \subset B(X)$, then there exists $v \in X$ such that $Sx_o = Bv$ and

$M(Ax_0, Bv, t) = K$. We have to prove that $K = 1$.

Suppose on the contrary that $K \neq 1$ then $K < 1$.

Putting $x_n = x_0$ and $y = v$ in (d) we get,

$F[\, M(Sx_o, Tv, t), M(Ax_0, Bv, t), M(Ax_o, Sx_o, t), M(Bv, Tv, t), M(Ax_o, Tv, t), M(Sx_o, Bv, t)\,] > 0$

$\Longrightarrow \quad F[\, M(Bv, Tv, t), K, K, M(Bv, Tv, t), M(Ax_o, Tv, t), 1\,] > 0$

By (C_a), we get

$$M(Bv, Tv, t) > K.$$

Since $T(X) \subset A(X)$, then there exists $u \in X$ such that $Tv = Au$ and

$M(Au, Bv, t) > K, \quad \forall\, t > 0.$

Since $M(Au, Su, t) \leq K > 0$ and putting $x = u, y = u$ in (d), we have

$F[\, M(Su, Tv, t), M(Au, Bv, t), M(Au, Su, t),$

$M(Bv, Tv, t), M(Au, Tv, t), M(Su, Bv, t)\,] > 0, \quad \forall\, t > 0,$

$\Longrightarrow F[\, M(Su, Au, t), M(Tv, Bv, t), M(Au, Su, t),$

$M(Bv, Tv, t), 1, M(Su, Bv\, t)\,] > 0, \quad \forall\, t > 0$

By (C_b) we get,

$$K \geq M(Au, Su, t) > M(Bv, Tv, t) > K$$

which is a contradiction.

Then $K = 1$ which implicites that $Ax_0 = Sx_0 = Bv$

If $M(Bv, Tv, t) < 1$ and putting $x = x_0$, $y = v$ in (d) ,we have

$F[\, M(Sx_0, Tv, t), M(Ax_0, Bv, t), M(Ax_0, Sx_0, t),$

$M(Bv, Tv, t), M(Ax_0, Tv, t), M(Sx_0, Bv, t)\,] > 0$

$\Longrightarrow F[\, M(Bv, Tv, t), 1, 1, M(Bv, Tv, t), M(Bv, Tv, t), 1\,] > 0$

which is a contracliction of (C_3) .

Therefore $M(Bv, Tv, t) = 1$.

we get $Bv = Tv = Sx_0 = Ax_0 = z$.

Since the pair (S, A) is weak-compatible , we get $Az = Sz$.

Now, we are going to show that $z = Sz$, i.e. $M(z, Sz, t) = 1$.

Suppose on the contrary that $M(z, Sz, t) \neq 1$, then $M(z, Sz, t) < 1$.

Putting $x = z, y = v$ in (d) we get

$$F[M(Sz, Tv, t), M(Az, Bv, t), M(Az, Sz, t), M(Bv, Tv, t), M(Az, Tv, t), M(Sz, Bv, t)] > 0$$

$$\Rightarrow \quad F[M(Sz, z, t), M(Sz, z, t), 1, 1, M(Sz, z, t), M(Sz, z, t)] > 0$$

which is a contradiction of (C_3)

Therefore $z = Sz = Az$. (1)

Again, since the pair (B, T) is weak-compatible, we get $Tz = Bz$.

For showing $z = Tz$, so suppose on the contrary that $z \neq Tz$.

Then $M(z, Tz, t) < 1$, using (d), we get

$$F[M(Sz, Tz, t), M(Az, Bz, t), M(Az, Sz, t), M(Bz, Tz, t), M(Az, Tz, t), M(Sz, Bz, t)] > 0$$

$$\Rightarrow \quad F[M(z, Tz, t), M(z, Tz, t), 1, 1, M(z, Tz, t), M(z, Tz, t)] > 0$$

which is a contradiction of (C_3).

Therefore $z = Bz = Tz$. (2)

From (1) and (2)

$$z = Sz = Az = Bz = Tz.$$

Hence z is a common fixed point of A, B, S and T.

Uniqueness

Let z' be another common fixed point of A, B, S and T, i.e.

$z' = Az' = Bz' = Sz' = Tz'$ and $M(z, z', t) < 1$

By putting $x = z, y = z'$ in (d), we have

$$F[M(Sz, Tz', t), M(Az, Bz', t), M(Az, Bz, t), M(Bz, Tz', t), M(Az, Tz', t), M(Sz, Bz', t)] > 0$$

$$\Rightarrow \quad F[M(z, z', t), M(z, z', t), 1, 1, M(z, z', t), M(z, z', t)] > 0$$

which is a contradiction of (C_3).

Hence z is the unique common fixed point of A, B, S and T.

1.2. Common Fixed Point Theorem In Compact Fuzzy 2-Metric Space Satisfying Implicit Relation

Definition 1.2.1. The mapping $*: [0, 1]^3 \longrightarrow [0, 1]$ is called **t- norm** if

(1) $*(a, 1, 1) = a$, for all $a \in [0, 1]$,

(2) $*(a, b, c) = *(a, c, b) = *(c, a, b)$,

(3) $*(a, b, c) \le *(d, e, f)$, whenever $a \le d, b \le e, c \le f$,

(4) $*[*(a, b, c), d, e] = *[a, *(b, c, d), e] = *[a, b, *(c, d, e)]$,

Definition1.2.2. The 3- tuple (X, M, *) is called a **fuzzy 2- metric space** if X is an arbitrary set, *is a continuous t-norm and M is a fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions, for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$

(1) $M(x, y, z, 0) = 0$,

(2) $M(x, y, z, t) = 1$, $t > 0$ and when at least of the three points are equal

(3) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$

(4) $M(x, y, z, t_1+ t_2+ t_3) \ge M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$,

(This corresponds to tetrahedral or rectangular inequality in 2-Metric space)

(5) $M(x, y, z, t) : (0, \infty) \to (0,1)$ is continuous,

(6) $\lim_{n\to\infty} M(x, y, z, t) = 1$,

The function value M(x, y, z, t) may be interpreted as the probability that the area of triangle is less than t.

Definition1. 2.3. Let (X, M, *) be a fuzzy 2-metric space. Then we define an **open ball** with centre $x_0 \in X$ and radius r, $0 < r < 1, t > 0$ as

$$B(x_0, r, t) = \{ y \in X : S(x_0, y, z, t) > 1- r \}.$$

Definition 1.2.4. Let (X, M,*) be a fuzzy 2-metric space. Define

$$\tau = \{A \subset X : x \in A \text{ if and only if there exist } t > 0 \text{ and}$$

$$0 < r < 1 \text{ such that } B(x_0, r, t) \subset A\}.$$

Then τ is a topology on X.

Definition 1.2.5. Let (X, M, *)be a fuzzy 2-metric space.Then collection C = $\{G_\alpha : \alpha \in \Lambda$; where G_α is open sets of X } is said to be a **open cover** of X. If $\bigcup_{\alpha\in\Lambda} G_\alpha = X$.

Definition 1.2.6. A fuzzy 2-metric space(X, M, *) is said to be **compact** if every open covering of X has a finite sub coving.

Definition 1.2.7. Let (X, M,*) is a fuzzy 2-metric space.

(1) A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be **convergent** to

a point $x \in X$, if $\lim_{n\to\infty} M(x_n, x, a, t) = 1$, for all $a \in X$ and $t > 0$.

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a **cauchy sequence**, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1, \text{ for all } a \in X \text{ and } t > 0, p > 0.$$

(3) A fuzzy 2-metric space, in which every cauchy sequence is convergent, is said to be **complete**.

Definition1. 2.8. A function M is **continuous** in fuzzy 2-metric space iff whenever $x_n \to x$, $y_n \to y$, then

$$\lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a, t) \ ; \text{ for all } a \in X \text{ and } t > 0$$

Definition 1.2.9 . Two mappings A and S on fuzzy 2-metric space are **compatible** if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, a, t) = 1 \ ; \ \forall a \in X, \ t > 0$$

Whenever, $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

Definition 1.2.10. Self mappings A and S of a fuzzy 2-metric space

(X, M, *) are said to be **weak-compatible** if they commute at their coincidences points i.e. if $Ax = Sx$ for some $x \in X$ then $SAx = ASx$.

Theorem 1.2.11. Let A, B, S and T be self mappings of a compact fuzzy 2- metric space (X, M,*) satisfying:

(a) $S(X) \subset B(X)$ and $T(X) \subset A(X)$,

(b) the pair (S, A) and (T, B) are weakly compatible,

(c) S and A are continuous,

(d) inequality

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www.ijarse.com
IJARSE
ISSN 2319 - 8354

$F\,[M(Sx, Ty, a, t), M(Ax, By, a, t), M(Ax, Sx, a, t),$

$M(By, Ty, a, t), M(Ax, Ty, a, t), M(Sx, By, a, t)] > 0,$

$\forall$ x, y a $\in X$ and $F \in F_6$ satisfies (C_1), (C_2) and (C_3) for which one of

$M(Ax, By, a, t)$, $M(Ax, Sx, a, t)$ and $M(By, Ty, a, t)$ is positive.Then A, B, S and T have a unique common fixed point in X.

Proof. Let $K = \text{Sup}\,\{M(Ax, Sx, a, t)\ ;\ x, a \in X,\ t > 0\}$.

Since X is compact, then there is a convergent sequence $\{x_n\}$ with limit $x_0 \in X$ such that

$$\lim_{n\to\infty} M(Ax_n, Sx_n, a, t) = K, \quad t > 0.$$

Since,

$$M(Ax_0, Sx_0, a, t) \geq M(Ax, Ax_n, Sx_n, t/9) * M(Ax_0, Sx_n, a, t/9) * M(Sx_n, Ax_n, a, t/9) * M(Ax_0, Sx_0, Sx_n, t/9) *$$
$$M(Sx_n, Sx_0, Ax_n, t/9) * M(Ax_n, Sx_0, Sx_n, t/9) * M(Sx_n, Sx_0, a, t/9).$$

By the continuity of A and S and $\lim_{n\to\infty} x_n = x_0$, we get,

$$M(Ax_0, Sx_0, a, t) \geq K \quad \forall\ t > 0,$$

Hence, $M(Ax_0, Sx_0, a, t) = K$. [by the definition of K].

Since $S(X) \subset B(X)$, then there exists $v \in X$ such that $Sx_0 = Bv$ and $M(Ax_0, Bv, a, t) = K$.

We have to prove that $K = 1$.

Suppose on the contrary that $K \neq 1$ then $K < 1$.

Putting $x = x_0$, $y = v$ in (d), we get,

$$F\,[M(Sx_0, Tv, a, t), M(Ax_0, Bv, a, t), M(Ax_0, Sx_0, a, t),$$
$$M(Bv, Tv, a, t), M(Ax_0, Tv, a, t), M(Sx_0, Bv, a, t)] > 0,$$

$$\Rightarrow \quad F\,[M(Bv, Tv, a, t), K, K, M(Bv, Tv, a, t), M(Ax_0, Tv, a, t), 1] > 0$$

by (C_a), we get

$$M(Bv, Tv, a, t) > K.$$

Since $T(X) \subset A(X)$ then there exists $u \in X$ such that $Tv = Au$ and

$M(Au, Bv, a, t) > K, \quad \forall\ t > 0.$

Since by the definition of K,

$M(Au, Su, a, t) \leq K > 0.$

Putting $x = u, y = v$ in (d), we obtain,

$$F [M(Su, Tv, a, t), M(Au, Bv, a, t), M(Au, Su, a, t)$$

$$M(Bv, Tv, a, t), M(Au, Tv, a, t), M(Su, Bv, a, t)] > 0$$

$$\Rightarrow \quad F [M(Au, Su, a, t), M(Bv, Tv, a, t), M(Au, Su, a, t),$$

$$M(Bv, Tv, a, t), 1, M(Su, Bv, a, t)] > 0 \ ; \quad \forall t > 0$$

By (C_b) we get,

$$K \geq M(Au, Su, a, t) > M(Bv, Tv, a, t) > K$$

which is a contradiction.

Then K =1 which implies that $Ax_0 = Sx_0 = Bv$.

If $M(Bv, Tv, a, t) < 1$ and Putting $x = x_0, y = v$ in (d) then we have

$$F [M(Sx_0, Tv, a, t), M(Ax_0, Bv, a, t), M(Ax_0, Sx_0, a, t)$$

$$M(Bv, Tv, a, t), M(Ax_0, Tv, a, t), M(Sx_0, Bv, a, t)] > 0$$

$$\Rightarrow \quad F [M(Bv, Tv, a, t), 1, 1, M(Bv, Tv, a, t), M(Bv, Tv, a, t) ,1] > 0$$

which is a contradiction of (C_3).

Therefore $M(Bv, Tv, a, t) = 1$, we get

$$Bv = Tv = Sx_0 = Ax_0 = z.$$

Since the pair (S, A) is weak-compatible, then $Az = Sz$.

Now, we are going to show that $z = Sz$ i.e. $M(z, Sz, a, t) = 1$.

Suppose on the contrary that $M(z, Sz, a, t) \neq 1$,then $M(z, Sz, a, t) < 1$.

Putting $x = z, y = v$ in (d), we get

$$F [M(Sz, Tv, a, t), M(Az, Bv, a, t), M(Az, Sz, a, t),$$

$$M(Bv, Tv, a, t), M(Az, Tv, a, t), M(Sz, Bv, a, t)] > 0$$

$$\Rightarrow F [M(Sz, z, a, t), M(Sz, z, a, t), 1, 1, M(Sz, z, a, t), M(Sz, z, a, t)] > 0$$

which is a contradiction of (C_3)

Therefore $z = Sz = Az.$ (1)

Again, since the pair (B, T) is weak-compatible, we get $Tz = Bz$. (2)

For showing $z = Tz$, suppose on the contrary that $z \neq Tz$,

then $M(z, Tz, a, t) < 1$. Putting $x = z, y = z$ in (d), we have

$$F [M(Sz, Tz, a, t), M(Az, Bz, a, t), M(Az, Sz, a, t),$$

$$M(Bz, Tz, a, t), M(Az, Tz, a, t), M(Sz, Bz, a, t)] > 0$$

$$\Rightarrow \quad F [M(z, Tz, a, t), M(z, Tz, a, t), 1, 1, M(z, Tz, t), M(z, Tz, t)] > 0$$

which is a contradiction of (C_3).

Therefore $z = Tz$. (3)

From (1) (2) and (3)

$$z = Bz = Tz = Az = Sz.$$

Hence z is a common fixed point of A, B, S and T.

Uniqueness

Let z' be another common fixed point of A, B, S and T

that is $z' = Az' = Bz' = Sz' = Tz'$ and $M(z, z', t) < 1$

By putting $x = z, y = z'$ in (d), we have

$$F [M(Sz, Tz', a, t), M(Az, Bz', a, t), M(Az, Bz, a, t),$$

$$M(Bz', Tz', a, t), M(Az, Tz', a, t), M(Sz, Bz', a, t)] > 0$$

$$\Rightarrow \quad F [M(z, z', a, t), M(z, z', a, t), 1, 1, M(z, z', t), M(z, z', a, t)] > 0$$

which is a contradiction of (C_3).

Hence z is a unique common fixed point of A, B, S and T.

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