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INTEGRAL REPRESENTATION OF NONRELATIVISTIC COLOUMB GREEN'S FUNCTION- A REVIEW

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ABSTRACT

Green's function is an impulse response which describe how the system will react to a single point source. There is a powerful method of Green's function to solve Quantum mechanical Coulomb problem. Its solution is commonly studied using Schrodinger equation. If one is able to represent a closed form expression for Coulomb Green's function one can immediately extract the energy spectrum as well as the wave functions

I INTRODUCTION

Green's function provides a powerful tool to solve differential equation with boundary conditions so that the problem has a unique solution. If one knows Green's function of a problem one can write down the solution in a closed form as an integral involving the Green's function and the inhomogeneous term appearing in the differential equation. The idea is to directly formulate the problem for Green's function by excluding the arbitrary inhomogeneous terms.

$$Ly(x) = f(x), \quad y(a) = y(b) = 0 \qquad (1)$$

where $L\,(x)$ is the differential operator, $y\,(x)$ is an unknown function and $f\,(x)$ is a known inhomogeneous term. Then the standard procedure is as follows. We have to find the Green's function $G(x, x')$ such that it obeys the differential equation

$$LG(x,x') = \delta(x - x') \qquad (2)$$

Green's function will satisfy the same boundary condition of the solution $y(x)$.

$$G(a|x') = G(b|x') = 0 \qquad (3)$$

Once we obtain the Green's function from the above conditions we can write the solution as

$$y(x) = \int_a^b G(x,x')\,f(x')\,dx' \qquad (4)$$

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II COULOMB GREEN'S FUNCTION FOR SCHRODINGER EQUATION

The Green's function G for the non relativistic Schrodinger equation is the solution of the equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{Z}{r} - E\right) G(\boldsymbol{r}, \boldsymbol{r}', E) = \delta(\boldsymbol{r} - \boldsymbol{r}') \qquad (5)$$

In spherical polar coordinates

$$\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\hbar^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\hbar^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] - \frac{Z}{r} - E\right) G(\boldsymbol{r}, \boldsymbol{r}', E) = \delta(\boldsymbol{r} - \boldsymbol{r}') \qquad (6)$$

this can be written as

$$\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2\hbar^2}L_{\theta,\varphi}\right] - \frac{Z}{r} - E\right) G(\boldsymbol{r}, \boldsymbol{r}', E) = \delta(\boldsymbol{r} - \boldsymbol{r}') \qquad (7)$$

Where $$L_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \qquad (8)$$

The partial wave expansion of $G(\boldsymbol{r}, \boldsymbol{r}', E)$ is

$$G(\boldsymbol{r}, \boldsymbol{r}', E) = \sum_{l,m} g_l(r, r')\, Y_{l,m}(\Omega) Y^*_{l,m}(\Omega) \qquad (9)$$

using *eq* : (15), *eq* : (6) becomes

$$\sum_{l,m}\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2\hbar^2}L^2{}_{\theta,\varphi}\right] - \frac{Z}{r} - E\right) g_l(r, r') Y_{l,m}(\Omega) Y^*_{l,m}(\Omega)$$

$$= \frac{1}{r^2\sin\theta}\delta(r - r')\delta(\theta - \theta')\delta(\varphi - \varphi') \qquad (10)$$

Using $$L^2{}_{\theta,\varphi} Y_{l,m}(\Omega) = l(l+1) Y_{l,m}(\Omega) \qquad (11)$$

equation (10) become

$$\sum_{l,m}\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2\hbar^2}\right] - \frac{Z}{r} - E\right) g_l(r, r') Y_{l,m}(\Omega) Y^*_{l,m}(\Omega)$$

$$= \frac{1}{r^2 \sin\theta}\delta(r - r')\delta(\theta - \theta')\delta(\varphi - \varphi') \qquad (12)$$

multiplying both side of the equation with $Y_{l',m'}(\Omega)$and integrating we get

$$\int \sum_{l,m}\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2\hbar^2}\right] - \frac{Z}{r} - E\right) g_l(r,r') Y_{l,m}(\Omega) Y^*_{l,m}(\Omega) Y_{l',m'}(\Omega)\, d\Omega$$

$$= \frac{\delta(r - r')}{rr'}\int \delta(\Omega - \Omega')\, Y_{l',m'}(\Omega) d\Omega \qquad (13)$$

From the orthogonality condition for spherical harmonics,

$$\int Y^*_{l,m}(\Omega) Y_{l',m'}(\Omega) d\Omega = \delta_{ll'}\delta_{mm'} \qquad (14)$$

$$\sum_{l,m}\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2}\right] - \frac{Z}{r} - E\right) g_l(r,r')\, Y_{l,m}(\Omega)\delta_{ll'}\delta_{mm'}$$

$$= \frac{\delta(r - r')}{rr'} Y_{l',m'}(\Omega') \qquad (15)$$

$$\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2}\right] - \frac{Z}{r} - E\right) g_{l'}(r,r') Y_{l',m'}(\Omega') = \frac{\delta(r - r')}{rr'} Y_{l',m'}(\Omega') \qquad (16)$$

Now separating out radial part we get

$$\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2}\right] - \frac{Z}{r} - E\right) g_l(r,r') = \frac{\delta(r - r')}{rr'} \qquad (17)$$

Now put $g_l(r,r') = \frac{G_l(r,r')}{rr'}$ Thus eq:(23) become

$$\left(-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2}\right] - \frac{Z}{r} - E\right) G_l(r,r') = \delta(r - r') \qquad (18)$$

Taking $G_l = \frac{u}{r}$we get the radial equation as

$$\frac{d^2u}{dr^2} + \left[\frac{2m}{\hbar^2}\left(E - \frac{Z}{r}\right) - \frac{l(l+1)}{r^2}\right]u = 0 \qquad (19)$$

Substituting $\frac{2mE}{\hbar^2} = p^2$, $\frac{Ze^2 m}{\hbar^2} = \gamma$, $\epsilon = \frac{\gamma}{p}$ and by taking x=ipr we get,

$$\frac{1}{p^2}\frac{d^2u}{dr^2} + \left(1 - \frac{2\epsilon}{pr} - \frac{l(l+1)}{p^2r^2}\right)u = 0 \qquad (20)$$

We get the non relativistic radial wave equation for a coulomb potential corresponding to angular momentum state *l* is in the form

$$\frac{d^2u}{dx^2} + \left(-1 + \frac{2i\epsilon}{x} - \frac{l(l+1)}{x^2}\right)u = 0 \qquad (21)$$

We will first consider solution to radial equation in this simple form, so that solutions will look more transparent.Equation (21) is similar to the Whittaker differential equation,which is of the form

$$\frac{d^2W}{dz^2} + \left(\frac{-1}{4} + \frac{k}{z} - \frac{\frac{1}{4} - m^2}{z^2}\right)W = 0 \qquad (22)$$

solution of the Whittaker differential equation are known as Whittaker functions, given as

$$W(Z) = \epsilon^{\frac{-z}{2}} z^{n+\frac{1}{2}} \frac{|\Gamma\left(m + \frac{1}{2} + k\right)|}{2m+1} {}_1^1F(-k + \frac{1}{2}, 2m+1, z) \qquad (23)$$

where ${}_1F_1$ is the confluent hyper geometric function of the first kind. Comparing eq:(27)and eq:(28) we can write $l + \frac{1}{2} = m$, $x = \frac{z}{2}$and k=iϵThe coulomb wave function can also be expressed in terms of Whittaker functions. Thus solution of eq:(21) is

$$u_1(x) = \frac{2^l x^l e^{-x}}{\Gamma(2l+2)} \Gamma(l + 1 - i\epsilon) {}_1^1F(l + 1 - i\epsilon, 2l + 2, 2x) \qquad (24)$$

This can be transformed into another integral which contain spherical Bessel function of integral order:

$$u_1(x) = \frac{x(2i)^{-l}}{\Gamma(i\epsilon)} \int_0^1 dt(1-t)^{i\epsilon - 1} t^{-i\epsilon} e^{-x(1-t)} j_l(ixt) \qquad (25)$$

We can prove this by proceeding from (25) to (24). We have

$$j_l(z) = \sqrt{\frac{\pi}{2z}} j_{l+\frac{1}{2}}(z) \qquad (26)$$

$$\frac{x}{(2i)^{-l}}\frac{1}{\Gamma(i\epsilon)}\int_0^1 dt(1-t)^{i\epsilon-1}t^{-i\epsilon}e^{-x(1-t)}j_l(ixt) =$$
$$= \frac{x}{(2i)^{-l}}\frac{e^{-x}}{\Gamma(i\epsilon)}\sqrt{\frac{\pi}{2ix}}\int_0^1 dt(1-t)^{i\epsilon-1}t^{-i\epsilon-\frac{1}{2}}e^{-xt}j_{l+\frac{1}{2}}(z) \qquad (27)$$

We have

$$\int_0^1 dt(1-t)^{\mu-1}t^{\lambda-\nu-1}e^{-i\alpha t}j_\nu(\alpha t) = \frac{\alpha^\nu}{2^\nu}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)\Gamma(\nu+1)}\,{}^2_2F\left(\lambda,\nu+\frac{1}{2},\lambda+\mu,2\nu+1,2x\right) \quad (28)$$

taking$ix = \propto$,$\nu = l+\frac{1}{2}$, $\mu = i\epsilon$, $\lambda = l+1-i\epsilon$ and substituting eq: (28) in eq: (27) it becomes,

$$\frac{x}{(2i)^{-l}}\frac{1}{\Gamma(i\epsilon)}\int_0^1 dt(1-t)^{i\epsilon-1}t^{-i\epsilon}e^{-x(1-t)}j_l(ixt)$$
$$= \frac{\sqrt{x}\sqrt{\pi}e^x}{(2i)^{l+\frac{1}{2}}}\frac{(ix)^{l+\frac{1}{2}}}{2^{l+\frac{1}{2}}}\frac{\Gamma(l+1-i\epsilon)\Gamma(i\epsilon)}{\Gamma(l+1)\Gamma\left(l+\frac{3}{2}\right)}\,{}^2_2F(l+1-i\epsilon\ ,l+1;l+1,2l+2,2x) \qquad (29)$$

$$= \frac{x^{l+1}e^{-x}}{2^{2l+1}}\frac{\Gamma(l+1-i\epsilon)}{\Gamma(l+1)\Gamma\left(l+\frac{3}{2}\right)}\sqrt{\pi}\,{}^1_1F(l+1-i\epsilon\ ,(2l+2);2x) \qquad (30)$$

Thus we have

$$u_1(x) = \frac{x(2i)^{-l}}{\Gamma(i\epsilon)}\int_0^1 dt(1-t)^{i\epsilon-1}t^{-i\epsilon}e^{-x(1-t)}j_l(ixt) \qquad (31)$$

we also have

$$\int_0^1 dt(1-t)^{i\epsilon-1}t^{-i\epsilon}\frac{\partial}{\partial t}\left[te^{-x(1-t)}j_l(ixt)\right] = \int_0^1 dt\left(\frac{1-t}{t}\right)^{i\epsilon-1}\frac{\partial}{\partial t}\left[te^{-x(1-t)}j_l(ixt)\right]$$

Integrating and by applying limit first term become zero, then we have

$$\int_0^1 dt(1-t)^{i\epsilon-1}t^{-i\epsilon}\frac{\partial}{\partial t}\left[te^{-x(1-t)}j_l(ixt)\right] = \int_0^1 i\epsilon\left(\frac{1-t}{t}\right)^{i\epsilon-1}\frac{1}{t}e^{-x(1-t)}j_l(ixt)dt \qquad (32)$$

and eq:(31) can be written as

$$u_1(x) = \frac{x(2i)^{-l}}{\Gamma(1+i\epsilon)}\int_0^1 dt(1-t)^{i\epsilon-1}t^{-i\epsilon}\frac{\partial}{\partial t}\left[te^{-x(1-t)}j_l(ixt)\right] \qquad (33)$$

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$u_1(x)$ as given by equation is an analytic function of ϵ inside the circle $|\epsilon| = 1$ for every l.

An analogous result for the second solution which is finite at infinity is derived in thefollowing.

The second solution of Whittakerdeferential equation can be expressed in terms of confluent hyper geometric function of second kind U(a;b;z) and its integral representation is

$$U(a;b;z) = \frac{1}{\Gamma a}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}dt \qquad (34)$$

we get the second solution as

$$u_2(x) = \frac{x^{l+1}e^{-x}}{\Gamma(l+1-i\epsilon)}\int_1^\infty ds(s-1)^{l-i\epsilon}s^{l+i\epsilon}e^{2xs} \qquad (35)$$

It can be written as

$$u_2(x) = \frac{-x(-2i)^l}{2\Gamma(-i\epsilon)}\int_1^\infty ds(s-1)^{-i\epsilon-1}s^{i\epsilon}h_l(ixs)\,e^{-x(s-1)} \qquad (36)$$

We can prove this by proceeding from eq:(43) to (42). On using modified Poisson integral representation

for $h_l(ixs)$, $u_2(x)$ as given by Eq.(36) can be written as follows

$$u_2(x) = \frac{x^{l+1}}{\Gamma(-i\epsilon)\Gamma(l+1)}\int_1^\infty ds(s-1)^{-i\epsilon-1}s^{l+i\epsilon}\int dp(p-1)^l p^l e^{x(1-2sp)} \qquad (37)$$

$$\int_1^\infty dp\frac{\partial}{\partial\beta}\frac{(p-1)^l e^{x(1-2sp\beta)}}{(2xs)^l}\Bigg|_{\beta=1} = \int_1^\infty dp(p-1)^l\,p^l e^{x(1-2sp)} \qquad (38)$$

Substituting eq:(38) in the eq.(37) we get

$$u_2(x) = \frac{x^{l+1}}{\Gamma(-i\epsilon)\Gamma(l+1)}\int ds\,(s-1)^{-i\epsilon-1}s^{l+i\epsilon}\int_1^\infty dp\left[\frac{\partial^l}{\partial\beta^l}(p-1)^l\frac{e^{x(1-2sp\beta)}}{(-2xs)^l}\right]_{\beta=1} \qquad (39)$$

On integrating by part with respect to p we get

$$\int_1^\infty dp\,\frac{\partial^l}{\partial\beta^l}(p-1)^l\,\frac{e^{x(1-2sp\beta)}}{(-2xs)^l} = \frac{\partial^l}{\partial\beta^l}\,l!\,\frac{e^{x(1-2s\beta)}}{(-2xs)^l(2xs\beta)^{l+1}} \qquad (40)$$

Thus, $\quad u_2(x) = \frac{x^{l+1}}{\Gamma(-i\epsilon)\Gamma(l+1)}\int_1^\infty\int ds\,(s-1)^{-i\epsilon-1}s^{l+i\epsilon}\,l!\,\frac{\partial^l}{\partial\beta^l}\frac{e^{x(1-2s\beta)}}{(-2xs)^l(2xs\beta)^{l+1}}$

$$= \frac{x^{l+1}}{\Gamma(-i\epsilon)} \int_1^\infty ds (s-1)^{-i\epsilon-1} s^{i\epsilon-l-1} \frac{\partial^l}{\partial \beta^l} \frac{e^{x(1-2s\beta)}}{(-2x)^l (2x\beta)^{l+1}} \qquad (41)$$

this can be written as

$$u_2(x) = \frac{x^{l+1}}{\Gamma(l+1-i\epsilon)} \int_1^\infty ds \frac{\Gamma(l+1-i\epsilon)}{\Gamma(-i\epsilon)} (s-1)^{-i\epsilon-1} s^{i\epsilon-l-1} \frac{\partial^l}{\partial \beta^l} \frac{e^{x(1-2s\beta)}}{(-2x)^l (2x\beta)^{l+1}} \qquad (42)$$

On making use of the identity

$$\frac{\partial^{l+1}}{\partial s^{l+1}} [s^{i\epsilon} (s-1)^{l-i\epsilon}] = \frac{\Gamma(l+1-i\epsilon)}{\Gamma(-i\epsilon)} (s-1)^{-i\epsilon-1} s^{i\epsilon-l-1} \qquad (43)$$

We get $u_2(x) = \frac{x^{l+1}}{\Gamma(l+1-i\epsilon)} \int_1^\infty ds \frac{\partial^{l+1}}{\partial s^{l+1}} [s^{i\epsilon}(s-1)^{l-i\epsilon}] \frac{\partial^l}{\partial \beta^l} \left(\frac{e^{x(1-2s\beta)}}{(-2x)^l (2x\beta)^{l+1}} \right)_{\beta=1}$ (44)

On integrating with respect to s we get,

$$u_2(x) = \frac{x^{l+1}}{\Gamma(l+1-i\epsilon)} \left[\frac{\partial^l}{\partial \beta^l} \left(\frac{e^{x(1-2s\beta)}}{(-2x)^l (2x\beta)^{l+1}} \right) \frac{\partial^l}{\partial s^l} [s^{i\epsilon}(s-1)^{l-i\epsilon}] \right]_{s=1}^{s=\infty} - \int_1^\infty \frac{\partial^l}{\partial \beta^l} \left(\frac{e^{x(1-2s\beta)}}{(-2x)^l (2x\beta)^{l+1}} \right) (-2x\beta) \frac{\partial^l}{\partial s^l} [s^{i\epsilon}(s-1)^{l-i\epsilon}] ds \qquad (45)$$

In the above equation first term will become zero by applying the limit. similarly partially integrating $l+1$ times.

$$u_2(x) = \frac{x^{l+1}}{\Gamma(l+1-i\epsilon)} \int_1^\infty \frac{\partial^l}{\partial \beta^l} \left(\frac{e^{x(1-2s\beta)}}{(-2x)^l (2x\beta)^{l+1}} \right) (-2x\beta)^{l+1} s^{i\epsilon} (s-1)^{l-i\epsilon} ds \qquad (46)$$

$$u_2(x) = \frac{x^{l+1}}{\Gamma(l+1-i\epsilon)} \int_1^\infty ds\, s^{i\epsilon} (s-1)^{l-i\epsilon} \left[\frac{\partial^l}{\partial \beta^l} \frac{e^{x(1-2s\beta)}}{(-2x)^l} \right]_{\beta=1} \qquad (47)$$

Differentiating eq:(47) l times with respect to β, we get

$$u_2(x) = \frac{x^{l+1}}{\Gamma(l+1-i\epsilon)} \int_1^\infty ds\, s^{l+i\epsilon} (s-1)^{l-i\epsilon} e^{-2xs} \qquad (48)$$

Thus we have

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$$u_2(x) = \frac{-x(-2i)^{-l}}{2\Gamma(-i\epsilon)} \int_1^\infty ds\,(s-1)^{-i\epsilon-1} s^{i\epsilon} h_l^1(ixs) e^{-x(s-1)} \qquad (49)$$

Using

$$\frac{-x(-2i)^{-l}}{2\Gamma(1-i\epsilon)} \int_1^\infty ds\,(s-1)^{-i\epsilon} s^{i\epsilon} \frac{\partial}{\partial s}\left[s e^{-x(s-1)} h_l^1(ixs)\right]$$
$$= \frac{-x(-2i)^{-l}}{2\Gamma(1-i\epsilon)} \int_1^\infty ds \left(\frac{s}{s-1}\right)^{i\epsilon} \frac{\partial}{\partial s}\left[s e^{-x(s-1)} h_l^1(ixs)\right] \qquad (50)$$

On integrating by part with respect to s eq:(50) can be written as

$$= \frac{-x(-2i)^{-l}}{2\Gamma(1-i\epsilon)}\left[\left(\frac{s}{s-1}\right)^{i\epsilon} s e^{-x(s-1)} h_l^1(ixs)\right] - \int_1^\infty i\epsilon \left(\frac{s}{s-1}\right)^{i\epsilon-1} \frac{1}{(s-1)^2} s e^{-x(s-1)} h_l^1(ixs)\, ds$$

$$= \frac{-x(-2i)^{-l}}{2\Gamma(-i\epsilon)} \int_1^\infty ds\, s^{i\epsilon} (s-1)^{-i\epsilon-1} h_l^1(ixs) e^{-x(s-1)} \qquad (51)$$

Therefore $\quad u_2(x) = \frac{-x(-2i)^{-l}}{2\Gamma(1-i\epsilon)} \int_1^\infty ds\, s^{i\epsilon}(s-1)^{-i\epsilon} \frac{\partial}{\partial s}\left[s e^{-x(s-1)} h_l^1(ixs)\right] \qquad (52)$

We now consider radial solution to the Schrodinger equation for a Coulomb potential, corresponding to angular momentum state *l*. They are usually denoted by $w_1(r)$and $w_2(r)$are related to $u_1(x)$and$u_2(x)$ by

$$w_1(r) = C_l u_1(x) \qquad (53)$$

$$w_2(r) = D_l u_2(x) \qquad (54)$$

where $C_l$and D_l are constant normalization factors . Since Green's function is constructed from a function

which is independent of their normalizations, we will for simplicity set$C_l = D_l = 1$. The Green's function

is obtained following the usual procedure as

Let$w_1(x)$ be a solution of this obeying the boundary condition $r = 0$ and let$w_2(x)$ be a solution of the same differential equation obeying the boundary condition at $r = \infty$

Then we can write the Green's function as

$$G_l(r', r'', z) = w_1(r'), \qquad r' > r''$$

$$= w_2(r'), \qquad r' < r''$$

To make $G_l(r', r'', z)$ continues at $r' = r''$ multiply the first expression by the value of second at $r' = r''$ and multiply second expression by the value the of first at $r' = r''$ Thus we get

Thus $$G_l(r', r'', z) = \frac{w_1(r')w_2(r'')}{j(z)}, \quad r' > r'' \tag{55}$$

$$= \frac{w_2(r')w_1(r'')}{j(z)}, \quad r' < r'' \tag{56}$$

the wronskian $$j(z) = w_1(r')w_2^{'(r)} - w_2(r')w_1{}'(r') \tag{57}$$

$$j(z) = ip2^{-2l-1} \tag{58}$$

Thus for $r'' < r'$ we have

$$G_l(r', r'', z) = \frac{w_2(r')w_1(r'')}{j(z)} \tag{59}$$

from the eq:(52) we can write

$$w_2(r') = \frac{-ipr'(-2i)^l}{2\Gamma(1-i\epsilon)} \int_1^\infty ds\, s^{i\epsilon}(s-1)^{-i\epsilon} \frac{\partial}{\partial s}\left[s e^{ipr'(s-1)} h_l^1(pr's)\right] \tag{60}$$

also from the equation eq:(33)

$$w_1(r'') = \frac{-ipr''(-2i)^l}{2\Gamma(1+i\epsilon)} \int_0^1 (1-t)^{i\epsilon} t^{-i\epsilon} \frac{\partial}{\partial t}\left[t e^{ipr''(1-t)} j_l(pr''t)\right] dt \tag{61}$$

substituting eq:(60) and eq:(61) in eq:(59), we get

$$G_l(r', r'', z) = \frac{-i^l}{ip2^{-2l}} \frac{(ip)^2 r'r'' i^{-2l} 2^{-2l-1}}{\Gamma(1-i\epsilon)\Gamma(1+i\epsilon)} \int_1^\infty ds \int_0^1 dt\, [s(1-t)]^{i\epsilon} [t(1-s)]^{-i\epsilon} \frac{\partial^2}{\partial t \partial s}\left[e^{ip[r''(1-t)-r'(1-s)]} st\, j_l(prt) h_l^1(prs)\right] \tag{62}$$

$$G_l(r', r'', z) = \frac{ipr'r''}{\Gamma(1-i\epsilon)\Gamma(1+i\epsilon)} \int_1^\infty ds \int_0^1 dt\, [s(1-t)]^{i\epsilon} [t(1-s)]^{-i\epsilon} \frac{\partial^2}{\partial t \partial s}\left[e^{ip[r''(1-t)-r'(1-s)]} st\, j_l(pr't) h_l^1(pr''s)\right] \tag{63}$$

We can obtain the three dimensional green's function by summing over the angular momentum state.

$$G(\boldsymbol{r'},\boldsymbol{r''},z) = \sum_l \sum_m \frac{g_l(r',r'',z)}{r'r''} Y_{lm}(\theta\varphi) Y^*_{lm}(\theta\varphi) \qquad (64)$$

Using $$P_l(\cos\theta) = \sum_m \frac{4\pi}{2l+1} Y_{lm}(\theta\varphi) Y^*_{lm}(\theta\varphi) \qquad (65)$$

Thus we get $$G(\boldsymbol{r'},\boldsymbol{r''},z) = \sum_l \sum_m (4\pi r' r'')^{-1}(2l+1)\, P_l(\cos\theta)\, G_l(r',r'',z) \qquad (66)$$

$$G(\boldsymbol{r'},\boldsymbol{r''},z) = \sum_l \frac{2l+1}{4\pi r'r''} P_l(\cos\theta) \frac{ipr'r''}{\Gamma(1+i\epsilon)\Gamma(1-i\epsilon)} \int_1^\infty ds \int_0^1 dt\, [s(1-t)]^{i\epsilon}[t(1 - s)]^{-i\epsilon} \frac{\partial^2}{\partial t \partial s}\left[e^{ip[r'(1-t)-r''(1-s)]} st\, j_l(pr't) h_l^1(pr''s)\right] \qquad (67)$$

Interchanging the order of summation and integration

$$G(\boldsymbol{r'},\boldsymbol{r''},z) = \frac{ip}{4\pi\Gamma(1+i\epsilon)\Gamma(1-i\epsilon)} \int_1^\infty ds \int_0^1 dt\, [s(1-t)]^{i\epsilon}[t(1-s)]^{-i\epsilon} \frac{\partial^2}{\partial t\partial s}\left[e^{ip[r''(1-t)-r'(1-s)]}\right] \sum_l (2l + 1) P_l(\cos\theta)\, j_l(pr't) h_l^1(pr''s)$$

Using the familiar expansion

$$\frac{e^{ip|sr''-tr'|}}{ip|sr''-tr'|} = \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta)\, j_l(pr't) h_l^1(pr''s) \qquad (68)$$

we get

$$G(\boldsymbol{r'},\boldsymbol{r''},z) = \frac{ip}{4\pi\Gamma(1+i\epsilon)\Gamma(1-i\epsilon)} \int_1^\infty ds \int_0^1 dt\, [s(1-t)]^{i\epsilon}[t(1 - s)]^{-i\epsilon} \frac{\partial^2}{\partial t\partial s}\left[st\, e^{ip[r''(1-t)-r'(1-s)]} \frac{e^{ip|sr''-tr'|}}{ip|sr''-tr'|}\right] \qquad (69)$$

$$G(\boldsymbol{r'},\boldsymbol{r''},z) = \frac{1}{4\pi\Gamma(1+i\epsilon)\Gamma(1-i\epsilon)} \int_1^\infty ds \int_0^1 dt\, [s(1-t)]^{i\epsilon}[t(1 - s)]^{-i\epsilon} \frac{\partial^2}{\partial t\partial s}\left[st|sr''-tr'|^{-1} e^{ip[r'(1-t)+r''(s-1)+|sr''-tr'|]}\right] \qquad (70)$$

Up to now we have treated the case where potential is attractive. In the case the potential is repulsive,the sign of ϵ must be reversed.

III CONCLUSION

Although the radial Green's function for the Schrodinger equation in a Coulomb field can be obtained in usual way in terms of two linearly independent solutions to the radial equation for a particular angular momentum state,the sum over angular momentum state does not seem to have been carried out. Here I reviewed a paper where the sum over angular momentum state is carried out and a "closed form" for the Green's function is obtained in terms of a double integral. The result may be useful for perturbation calculations where the intermediate states involve many angular momentum state.

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