International Journal of Advance Research in Science and Engineering
Vol. No. 5, Special Issue No. 01 , May 2016
www.ijarse.com
IJARSE
ISSN 2319 - 8354

A STUDY OF PARTIAL DERIVATIVES AND ITS GEOMETRIC INTERPRETATION AND THEIR APPLICATIONS

Dr .Vasudevarao.kota

Assistant Professor, Department of Mathematics, Ambo University

I INTRODUCTION

In mathematics, a partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant (as opposed to the total derivative, in which all variables are allowed to vary). Partial derivatives are used in vector calculus and differential geometry

Since in general a partial derivative is a function of the same arguments as was the original function, this functional dependence is sometimes explicitly included in the notation, as in the partial-derivative symbol is ∂. One of the first known uses of the symbol in mathematics is by Marquis de Condorcet from 1770, who used it for partial differences. The modern partial derivative notation is by Adrien-Marie Legendre (1786), though he later abandoned it; Carl Gustav Jacob Jacobi re-introduced the symbol in 1841

II PARTIAL DERIVATIVES

Definition: Let f be a function of two variables, and let (x_0, y_0) be in the domain of f. The partial derivative of f with respect to x at (x_0, y_0) is defined by

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

provided that this limit exists. The partial derivative of f with respect to y at (x_0, y_0) is defined by

$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

provided that this limit exists.

Key Words: Partial Derivate, Gradient, Tangent Approximations, Higher Order Derivates, Chain Rules

Note: If $z = f(x, y)$, then we can write $f_x(x, y) = \frac{\partial z}{\partial x}$ and $f_y(x, y) = \frac{\partial z}{\partial y}$.

Example 1: Let $f(x, y) = 24xy - 6x^2y$.

Find f_x and f_y and evaluate f_x and f_y at $(1,2)$.

Solution: By holding y constant and differentiating f with respect of x, we find that

$f_x(x,y) = 24y - 12xy$ so that $f_x(1{,}2) = 48 - 24 = 24$. By holding x constant and differentiating f with respect to y, we find that $f_y(x,y) = 24x - 6x^2$ so that

$f_y(1{,}2) = 24 - 6 = \underline{\underline{18}}$.

Remark: $f_x(x_0, y_0)$ is slope or rate of change in x −direction at (x_0, y_0), and $f_y(x_0, y_0)$ is the slope of the surface in the y − direction at (x_0, y_0).

Example-2: Let $f(x,y) = x^2y + 5y^3$.

(a) Find the slope of the surface $z = f(x,y)$ in the x – direction at the point $(1,-2)$

(b) Find the slope of the surface $z = f(x,y)$ in the y – direction at the point $(1,-2)$

Solution: (a) Differentiating f with respect to with y held fixed yields $f_x(x,y) = 2xy$.

Thus, the slope in the x – direction is $f_x(1,-2) = -4$; that is, Z is decreasing at the rate of 4 units per unit increase in x.

(b) Differentiating f with respect to y with x held fixed yields $f_y(x,y) = x^2 + 15y^2$.

Thus, the slope in the y −direction is $f_y(1,-2) = 61$; that is, z is increasing at the rate of 61 units per unit increase in y.

2.1 A Geometric Interpretation of Partial Derivatives

When we hold y equal to a constant $y = y_0$, $z = f(x,y)$ becomes the function $z = f(x, y_0)$ of x, whose graph is the intersection of the surface $z = f(x,y)$ with the vertical plane $y = y_0$ (Figure 4). The x −derivative $f_x(x_0, y_0)$ is the slope in the positive x −direction of the tangent line to this curve at $x = x_0$.

Similarly, when we hold x equal to a constant x_0, $z = f(x,y)$ becomes the function

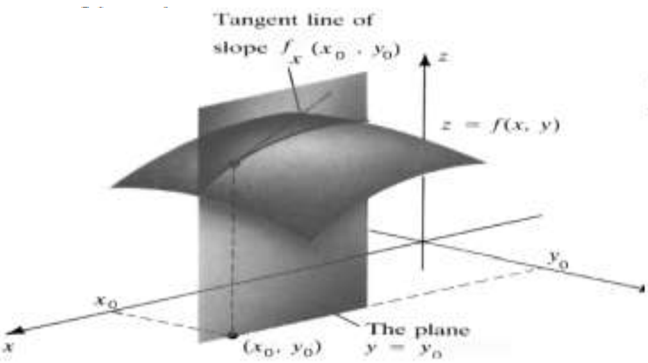


FIGURE 4

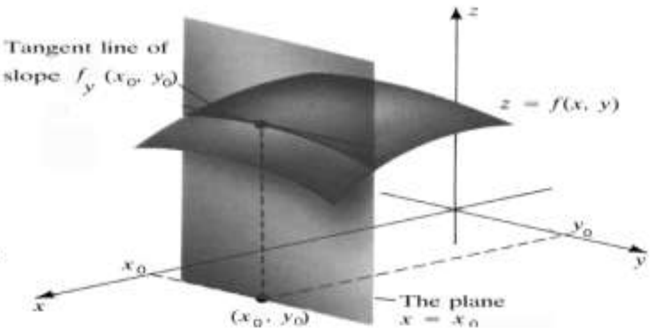


FIGURE 5

If $z = f\ (x, y)$, we use the following notation:

1. $$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

2. $$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

3. $$(f_y)x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

4. $$(f_y)y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation $f_{xy}\left(or\ \frac{\partial^2 f}{\partial y \partial x}\right)$ means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} there order is reversed.

Example-3: Find the second partial derivatives of $f(x, y) = \sin xy^2$.

Solution: The first partials are given by

$$f_x(x, y) = y^2 \cos xy^2\ and\ f_y(x, y) = 2xy\ cos\ xy^2$$

We obtain the second partials by computing the partial derivatives of the first partials:

$$f_{xx}(x, y) = -y^4 \sin xy^2$$

$$f_{xy}(x, y) = 2y \cos xy^2 - 2xy^2 \sin xy^2$$

$$f_{yx}(x, y) = 2y \cos y^2 - 2xy^3 \sin xy^2$$

$$f_{yy}(x, y) = 2y \cos xy^2 - 4x^2 y^2 \sin xy^2$$

Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Example-4: Calculate $f_{xxyz}\ if\ f\ (x, y, z) = \sin(3x + yz)$.

Solution: $f_x = 3\cos(3x + yz)$ $\quad f_{xx} = -9\sin(3x + yz)$ $\quad f_{xxy} = -9z\sin(3x + yz)$

$$f_{xxyz} = -9yz\sin(3x + yz) - 9\cos(3x + yz)$$

2.2 Differentiability of functions of several variables

Recall that in the case of a function of a single variable, a function $f(x)$ is differentiable only if it is continuous; but that continuity does not guarantee differentiability. Intuitively, continuity of $f(x)$ requires that its graph be a continuous curve; and differentiability requires also that there is always a unique tangent vector to the graph of $f(x)$. In other words, a function $f(x)$ is differentiable if and only if its graph is a smooth continuous curve with no sharp corners (a sharp corner would be a place where there would be two possible tangent vectors).

If we try to extend this graphical picture of differentiability to functions of two or more variables, it would be natural to think of a differentiable function of several variables as one whose graph is a smooth continuous surface, with no sharp peaks or folds. Because for such a surface it would always be possible to associate a unique tangent plane at a given point.

However, "differentiability" in this sense turns out to be a much stronger condition than the mere existence of partial derivatives. For the existence of partial derivatives at a point x_0 requires only a smooth approach to the point $f(x_0)$ along the direction of the coordinate axes. We have seen examples of functions that are discontinuous even though

$\lim_{x\to 0}(x,0) = \lim_{y\to 0}(0,y)$ both exist.

For example, the function $f(x,y) = \frac{(x-y)^2}{x^2+y^2}$ has this property, and in fact both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous functions at the point $(0,0)$.

With this sort of phenomenon in mind we give the following definition of differentiability.

Definition: We say that a function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ of two variables x and y is differentiable at (x_0, y_0) if

1. Both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$ exist at the point (x_0, y_0)
2. $\lim_{(x,y)} \frac{f(x,y)-f(x_0,y_0)-\left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0)-\left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

Remark: The limit condition simply means that

$$F(x,y) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y - y_0)$$

is a good approximation to $f(x, y)$ near the point (x_0, y_0).

The chain Rule

Recall that the chain rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Theorem (**The chain rule: case -1**)

Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Proof: A change of $\Delta t\ in\ t$ produces changes of $\Delta x\ in\ x\ and\ \Delta y\ in\ y$. Thus, in turn, produce a change of $\Delta z\ in\ z$, and we have $\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$

where $\epsilon_1 \to 0\ and\ \epsilon_2 \to 0\ as\ (\Delta x, \Delta y) \to (0,0)$. [If the functions $\epsilon_1\ and\ \epsilon_2$ are not defined at $(0,0)$, we can define them to be 0 there.] Dividing both sides of this equation by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}$$

If we not let $\Delta t \to 0$, then $\Delta x = g(t+\Delta t) - g(t) \longrightarrow 0$ because g is differentiable and therefore continuous.

Similarly, $\Delta y \longrightarrow 0$. This, in turn, means that $\epsilon_1 \longrightarrow 0\ and\ \epsilon_2 \longrightarrow 0$, so

$$\frac{dz}{dt} = \lim_{\Delta t\to 0}\frac{dz}{dt}$$

$$= \frac{\partial f}{\partial x}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t} + \left(\lim_{\Delta t\to 0}\epsilon_1\right)\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \left(\lim_{\Delta t\to 0}\epsilon_2\right)\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0.\frac{dx}{dt} + 0.\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Since we often write $\frac{dz}{dt}$ in place of $\frac{\partial f}{\partial x}$, we can rewrite the chain rule in the form:

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example-1: If $z = x^2y + 3xy^4, where\ x = sin2t\ and\ y = cost, find\ \frac{dz}{dt}\ \ when\ t = 0.$

Solution: The chain rule gives

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy + 3y^4)\ (2\cos 2t) + (x^2 + 12xy^3)\,(-sint)$$

Observe that when $t = 0$, we have $x = sin0 = 0\ \&\ y = \cos 0 = 1$.

Therefore $\left.\frac{dz}{dt}\right|_{t=0} = (0+3)\,(2\cos 0) + \ (0+0)\,(-\sin 0) = 6.$

Example-2: The pressure P (in kilopascal), volume V (in liter), and temperature T (in Kelvin) of a mole of an ideal gas are related by the equation PV = 8.31 T. Find the rate at which the pressure is changing when the temperature is 300k and increasing at a rate of 0.1 k/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

Solution: If t represents the time elapsed in seconds, then at given instant we have

$\mathrm{T} = 300,\ \frac{d\mathrm{T}}{dt} = 0.1,\ V = 100, \frac{dV}{dt} = 0.2$. Since $\mathrm{P} = 8.31\ \frac{\mathrm{T}}{V}$ the chain rule gives

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155$$

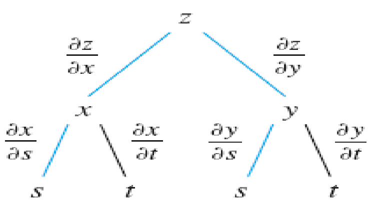
The pressure is decreasing at a rate of about $0.042\ kpa/s$.

We now consider the situation where $z = f(x,y)$ but each of $x \text{ and } y$ is a function of two variables $s \text{ and } t$: $x = g(s,t), y = h(s,t)$. Then Z is indirectly a function of $s \text{ and } t$ and we wish to find $\frac{\partial z}{\partial s} \text{ and } \frac{\partial z}{\partial t}$.

Theorem: (**The chain rule: case-2**)

Suppose that $z = f(x,y)$ is a differentiable function of $x \text{ and } y$, where $x = z(s,t)$ and $y = h(s,t)$. Then z is indirectly a function of $s \text{ and } t$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\ ,\ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Example-3: If z= $e^x \sin y, where\ x = st^2 \text{ and } y = s^2 t, find\ \frac{\partial z}{\partial s} \text{ and } \frac{\partial x}{\partial t}$.

Solution: Applying case 2 of the chain rule, we get

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)\,(t^2) + (e^x \cos y)\,(2st)$$

$$= t^2 e^{st^2}\sin(s^2 t) + 2ste^{st^2}\cos(s^2 t)$$

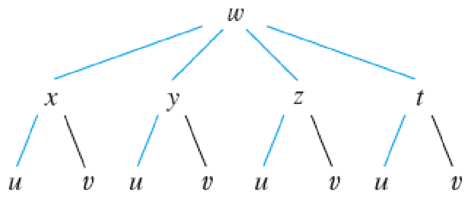
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)\,(2st) + (e^x \cos y)\,(s^2)$$

$$= 2ste^{st^2}\sin(s^2 t) + s^2 e^{st^2}\cos(s^2 t)$$

Example-4: Write out the chain rule for the case where $w = f(x,y,z,t)$ and

$x = x(u,v),\ \ y = y(u,v),\ \ \ \ z = z(u,v),\ \&\ \ t = t(u,v)$.

Solution: Using the tree diagram given below, we can write the required expressions.

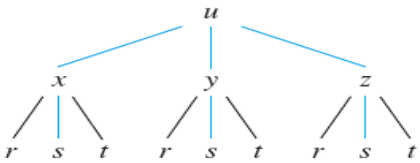


$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

Example-5: If $u = x^4y + y^2z^2, where\ x = rse^t, y = rs^2e^{-t}, and\ z = r^2s\sin t,$ find the value of $\frac{\partial u}{\partial s}$ when $r = 2,\ \ s = 1,\ t = 0.$

Solution:



We the help of the tree diagram given above, we have

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3y)\ (re^t) + (x^4 + 2yz^3)\ (2rse^{-t}) + (3y^2z^2\ )\ (r^2\sin t)$$

When $r = 2,\ \ s = 1,$ and $t = 0,$ we have$x = 2, y = 2, and\ z = 0,$

So $\frac{\partial u}{\partial s} = (64)\ (2) + \ (16)\ (4) + \ (0)\ (0) = 192.$

Example-6: If $z = f\ (x,y)$ has continuous second–order partial derivatives and

$x = r^2 + s^2$ and $y = 2rs,$ find

(a) $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial r^2}$

Solution:

(a) The Chain Rule gives

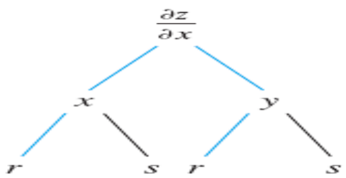
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

(b) Applying the product rule to the expression in part (a), we get

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(2r\,\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)$$

$$= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\ \ldots(*)$$

But, using the chain rule again, we have



$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y\partial k}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)$$

Putting these expressions into equation $(*)$ and using the equality of the mixed second order derivatives, we obtain

$$\frac{\partial^2 z}{\partial r^2} = 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y\partial x}\right) + 2s\left(2r\frac{\partial^2 z}{\partial x\partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)$$

$$= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x\partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}$$

Implicit Differentiation: If F is differentiable, we can apply case 1 of the chain rule to differentiate both sides of the equation $F(x,y) = 0$ with respect to x given that $y = f(x)$. Since both x and y are functions of x, we obtain.

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

But $\frac{dx}{dx} = 1,\ so\ if\ \frac{\partial F}{\partial y} \neq 0$ we solve for $\frac{dy}{dx}$ and obtain

$$\frac{dx}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}$$

Example 7 Find $y'\ if\ x^3 + y^3 = 6xy$.

Solution: The given equation can be written as

$$F(x,y) = x^3 + y^3 - 6xy = 0$$

So the above formula gives

$$\frac{dy}{dx} = -\frac{-F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2x}{y^2 - 2x}$$

Now we suppose that z is given implicitly as a function $z = f(x,y)$ by an equation of the form $F(x,y,z) = 0$. IF F and f are differentiable, then we can use the chain rule to differentiate the equation $F(x,y,z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But $\frac{\partial}{\partial x}(x) = 1\ and\ \frac{\partial}{\partial x}(y) = 0$ so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial^2}{\partial x} = 0$$

If $\frac{\partial F}{\partial z} \neq 0$, we solve for $\frac{\partial z}{\partial x}$ and obtain the first formula of the formulas given below. The formula for $\frac{\partial z}{\partial y}$ is obtained in a similar manner.

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}},\ \frac{\partial z}{\partial y} = -\frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example-8: Find $\frac{\partial z}{\partial x}\ and\ \frac{\partial z}{\partial y}\ if\ x^3 + y^3 + z^3 + 6xyz = 1$.

Solution: Let $F(x,y,z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then , from the above formulas, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy},\quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2+6yz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$$

Application of Partial Derivatives:

Directional Derivatives and Gradient of functions of several variables

Directional Derivative:

Definition: Let f be a function defined on a set containing a disk D centered at (x_0, y_0), and let $u = a_1 i + a_2 j$ be a unit vector. Then the **directional derivative** of f at (x_0, y_0) in the direction of u, denoted $D_u f\ (x_0, y_0)$, is defined by

$$D_u f\ (x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h a_1,\ y_0 + h a_2\ ) - f(x_0,\ y_0)}{h}$$

provided the limit exists.

Theorem: Let f be differentiable at (x_0, y_0). Then f has a directional derivative at (x_0, y_0) in every direction.

Moreover, if $u = a_1 i + a_2 j$ is a unit vector, then

$D_u f\ (x_0, y_0) = f_x(x_0, y_0) a_1 + f_y(x_0, y_0) a_2.$

Proof: Let $\mathrm{F}(h) = f(x_0 + h a_1, y_0 + h a_2)$

Then $\frac{\mathrm{F}(h) - \mathrm{F}(0)}{h - 0} = \frac{f(x_0 + h a_1,\ y_0 + h a_2\ ) - f(x_0,\ y_0)}{h}$.

So that $D_u f\ (x_0, y_0)$ exists if and only if $\mathrm{F}'(0)$ exists. If we let

$g_1(h) = x_0 + h a_1\ \ and\ g_2(h) = y_0 + h a_2$, then

$\mathrm{F}(h) = f(g_1(h), g_2(h)), and\ g_1(0) = x_0\ and\ g_2(0) = y_0.$

With h replacing t and 0 replacing to, the hypothesis of chain rule $\mathrm{F}'(0)$ exists, and

$$D_u f\ (x_0, y_0) = \mathrm{F}'(0) = f_x(x_0, y_0) g_1'(0) + f_y(x_0, y_0) g_2'(0)$$
$$= f_x(x_0, y_0) a_1 + f_y(x_0, y_0) a_2.$$

Example -1: Let $f\ (x, y) = 6 - 3x^2 - y^2$, and let $u = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$.

Find $D_u f\ (1,2)$.

Solution: Notice that u is a unit vector. First we calculate the partial derivatives of f:

$$f_x(x, y) = -6x\ \ and\ \ f_y(x, y) = -2y.$$

Therefore$D_u f\ (1,2) = f_x(1,2)\left(\frac{1}{\sqrt{2}}\right) + f_y(1,2)\left(\frac{-1}{\sqrt{2}}\right) = (-6)\ \left(\frac{1}{\sqrt{2}}\right) + (-4)\left(\frac{-1}{\sqrt{2}}\right) = -\sqrt{2}$

Remark: The directional derivative in the direction of an arbitrary non-zero vector a is defined to be $D_u f\ (x_0, y_0)$, where $u = \frac{1}{\|a\|} a$.

Example-2: Let $f\ (x, y) = xy^2$ and let $a = i - 2j$. Find the directional derivative at $(-3,1)$ in the direction of a.

Solution: In this case $\|a\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$, so we will find $D_u f(-3,1)$, where

$u = \frac{1}{\|a\|} a = \frac{1}{\sqrt{5}} i - \frac{2}{\sqrt{5}} j$. Since $f_x(x,y) = y^2$ and $f_y(x,y) = 2xy$

Thus

$$D_u f(-3,1) = f_x(-3,1)\left(\frac{1}{\sqrt{5}}\right) + f_y(-3,1)\left(\frac{-2}{\sqrt{5}}\right) = 1\left(\frac{1}{\sqrt{5}}\right) + (-6)\left(\frac{-2}{\sqrt{5}}\right) = \frac{13}{5}\sqrt{5}.$$

Let $u = a_1 i + a_2 j + a_3$ to be a unit vector in space. The directional derivative $D_u f\ (x_0, y_0, z_0)$ is defined by

$$D_u f\ (x_0, y_0, z_0) = \lim_{h \to 0} \frac{f\ (x_0 + ha_1, y_0 + ha_2, z_0 + ha_3) - f(x_0, y_0, z_0)}{h}$$

provided that the limit exists.

$$D_u f\ (x_0, y_0, z_0) = f_x(x_0, y_0, z_0)a_1 + f_y(x_0, y_0, z_0)a_2 + f_3(x_0, y_0, z_0)a_3$$

The Gradient

Definitions:

a) Let f be a function of two variables that has partial derivatives at (x_0, y_0). Then the **gradient** of f at (x_0, y_0), denoted grad $f(x_0, y_0)$ or $\nabla f(x_0, y_0)$ is defined by

$grad\ f(x_0, y_0) = \nabla f(x_0, y_0) = f_x(x_0, y_0)i + f_y(x_0, y_0)j$

b) Let f be a function of three variables that has partial derivatives at (x_0, y_0, z_0). Then the gradient of f at $(x_0, y_0, \mathfrak{z}_0)$, which is denoted $grad(x_0, y_0, \mathfrak{z}_0)$ or $\nabla f(x_0, y_0, z_0)$ is defined by

$$grad\ f(x_0, y_0, \mathfrak{z}_0) = \nabla f(x_0, y_0, z_0)$$
$$= f_x(x_0, y_0, \mathfrak{z}_0)i + f_y(x_0, y_0, z_0)j + f_3(x_0, y_0, z_0)\mathrm{k}$$

Example -3: Find the gradient of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$.

Solution: We first compute the partial derivatives at $(2,-1)$,

$f_x(x,y) = 2xy^2$ and $f_y(x,y) = 3x^2y^2 - 4.$

Hence $f_x(2,-1) = -4$ and $f_y(2,-1) = 8$

Therefore, $\nabla f(2,-1) = -4\,i + 8j$.

Notes: $D_u f\ (x,y) = \nabla f\ (x,y).\mathbf{u}$

Example-4: If $f(x,y,z) = x \sin yz$,

(a) Find the gradient of f and

(b) Find the directional derivative of f at $(1,3,0)$ in the direction of $v = i + 2j - k$.

Solution:

(a) The gradient of f is

$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z\,(x,y,z)\,\rangle = \langle \sin y\,z\,, xz\cos yz, xy\cos yz\rangle =$ $\langle 0,0,3\rangle$

(b) The unit vector in the direction of $v = i + 2j - k$ is $u = \frac{1}{\sqrt{6}}i + \frac{2}{\sqrt{6}}j - \frac{1}{\sqrt{6}}k$.

Therefore,

$$D_u f\,(1,3,0) = \nabla f(1,3,0)\cdot u = 3k\cdot\left(\frac{1}{\sqrt{6}}i + \frac{2}{\sqrt{6}}j - \frac{1}{\sqrt{6}}k\right)$$

$$= 3\left(\frac{-1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}}$$

Theorem: Suppose f is differentiable function of two or three variables. The maximum value of the directional derivative $D_u f\,(x)$ is $|\nabla f(x)|$ and it occurs when u has the same direction as the gradient vector $\nabla f(x)$.

Proof: We know that $D_u f = \nabla f\cdot u$

Thus, $D_u f = \nabla f\cdot u = |\nabla f|\,|u|\cos\theta = |\nabla f|\cos\theta$ where θ is the angle between ∇f *and* u. The maximum value of $\cos\theta$ *is* 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_u f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when u has the same direction∇f.

Example-5:

(a) If $f\,(x,y) = xe^y$, find the rate of change of f at the point $\mathrm{P}(2,0)$ in the direction from P to Q $\left(\frac{1}{2},2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Solution:

(a) We first compute the gradient vector:

$\nabla f(x,y) = \langle f_x, f_y\rangle = \langle e^y, xe^y\rangle \Rightarrow \nabla f(2,0) = \langle 1,2\rangle$

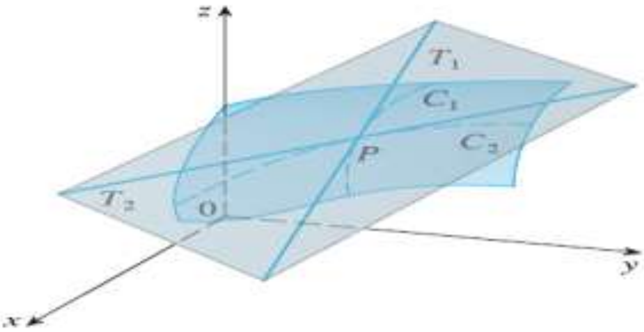
The unit vector in the direction of $\overrightarrow{PQ} = (-1{:}5,2)$ is a $u = \langle\frac{-3}{5},\frac{4}{5}\rangle$, so the rate of change of f in the direction from P to Q is

$$D_u f(2,0) = \nabla f(2,0) \cdot u = \langle 1,2 \rangle \cdot \langle \tfrac{-3}{5}, \tfrac{4}{5} \rangle = 1\left(\tfrac{-3}{5}\right) + 2\left(\tfrac{4}{5}\right) = \underline{\underline{1}}$$

(b) By the above Theorem f increases fastest in the direction of the gradient vector $\nabla f(2,0) = \langle 1,2 \rangle$. The maximum rate of change is $\langle \nabla f(2,0) \rangle = |\langle 1,2 \rangle| = \sqrt{5}$.

Tangent planes

Suppose a surface S has equation $z = f(x,y), where\ f$ has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. Let $C_1\ and\ C_2$ be the curves obtained by intersecting the vertical planes $y = y_0\ and\ x = x_0$ with the surface S. Then the point P lies on both $C_1\ and\ C_2$. Let $T_1\ and\ T_2$ be the tangent lines to the curves $C_1\ and\ C_2$ at the point P. Then the tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines $T_1\ and\ T_2$.



The tangent plane contains lines $T_1\ and\ T_2$

If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane. The tangent plane at P is the plane that most closely approximates surface S near the point P.

Any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form.

$$A(x - x_0) + B(y - y_0) + C\,(z - z_0) = 0$$

By dividing this equation by C and letting $a = \frac{-A}{C}$ and $b = \frac{-B}{C}$, we can write it in the form

$$Z - Z_0 = a(x - x_0) + b(y - y_0) \cdots (*)$$

$(*)$ represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in $(*)$ gives

$$Z - Z_0 = a(x - x_0) \qquad y = y_0$$

and we recognize these as the equations (in point – slope form) of a line with slope a.

But the slope of the tangent T_1 is $f_x\,(x_0, y_0)$. Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in $(*)$, we get $Z - Z_0 = b(y - y_0)$, which must represent the tangent line T_2, so $b = f_y(x_0, y_0)$.

Definition: Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $Z = f(x, y)$ at the point $P\,(x_0, y_0, z_0)$ is

$$Z - Z_0 = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example-1: Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.

Solution: Let $f(x, y) = 2x^2 + y^2$. Then $f_x\,(x, y) = 4x \quad f_y(x, y) = 2y$

$\Rightarrow f_x(1,1) = 4 \qquad f_y\,(1,1) = 2$

Then (by definition) the equation of the tangent plane at $(1,1,3)$ is

$$z - 3 = 4\,(x - 1) + 2\,(y - 1) \Rightarrow z = 4x + 2y - 3.$$

Tangent plane approximations and Differentials

Tangent plane approximations

In example 1 the tangent line $\mathcal{L}(x, y) = 4x + 2y - 3$ is a good approximation to $f(x, y)$ when (x, y) is near $(1,1)$.

The function L is called the linearization of f at $(1,1)$ and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the linear approximation or tangent plane approximation of f at $(1,1)$.

Definition: An equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is $z = f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b)$

is called the linearization of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b)$$

is called the linear approximation or the **tangent plane approximation** of f at (a, b).

Theorem: If the partial derivatives f_x *and* f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 2: Show that $f(x, y) = xe^{xy}$ is differentiable at $(1,0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution: The partial derivatives are

$f_x(x, y) = e^{xy} + xye^{xy}$ $\quad f_y\,(x, y) = x^2\, e^{xy} \Rightarrow f_x(1,0) = 1\ \&\ f_y\,(1,0) = 1$

Both f_x *and* f_y are continuous functions, so f is differentiable by the above theorem. The linearization is L $(x, y) = f(1,0) + f_x(1,0)\,(x - 1) + f_y(1,0)\,(y - 0) = x + y$

The corresponding linear approximation is $xe^{xy} \approx x + y$

So $f\,(1.1, -0.1) \approx 1.1 - 0.1 = 1$

Compare this with the actual value of $(1.1, -0.1) = 1.1e^{-0.1} \approx 0.98542$.

Differentials

If f is a function of two variables, we can replace (x_0, y_0) by any point (x, y) in the domain of f at which f is differentiable and the linear approximation is transformed into

$$f(x + h, y + k) - f\,(x, y) \approx f_x(x, y)h + f_y(x, y)k \cdots (*)$$

The number $f_x(x, y)\,h + f_y(x, y)k$ on the right side of $(*)$ is usually called the differential (or total differential) of f (at (x, y) with increments h *and* k) and is denoted df. Thus $= f_x(x, y)\,h + f_y(x, y)k \cdots\cdots (**)$. Of course, df depends on x, y, h *and* k, even though they are not indicated in the notation df.

If $g_1(x, y) = x$ *and* $g_2(x, y) = y$, then the differential dg_1 is denoted by dx, and the differential dg_2 is denoted by dy. Since

$(g_1)_x(x, y) = 1, (g_1)_y(x, y) = 0, (g_2)_x(x, y) = 0, (g_2)_y(x, y) = 1$

We have $dx = dg_1 = 1.h + 0.k = h$ *and* $dy = dg_2 = 0.h + 1.k = k$

Therefore we can write $(**)$ as

$$df = f_x(x,y)dx + f_y(x,y)dy \text{ or } df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Example -3: Let $f(x,y) = xy^2 + ysinx$. Find df.

Solution: Since $\frac{\partial f}{\partial x} = y^2 + y\cos x$ and $\frac{\partial f}{\partial y} = 2xy + sinx$

Thus $df = (y^2 + y\cos x)dx + (2xy + sinx)dy$

Note: If f is a function of three variables that is differentiable at $(x_{0,}y_0, z_0)$, then the differential df is defined by $df = f_x(x,y,z)\, \hbar + f_y(x,y,z)k + f_z(x,y,z)\ell$

The more usual form for df is

$$df = f_x(x,y,z)dx + f_y(x,y,z) + f_z(x,y,z)dz$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

.

Image Resizing

Partial derivatives are key to target-aware image resizing algorithms. Widely known as seam carving, these algorithms require each pixel in an image to be assigned a numerical 'energy' to describe their dissimilarity against orthogonal adjacent pixels. The algorithm then progressively removes rows or columns with the lowest energy. The formula established to determine a pixel's energy (magnitude of gradient at a pixel) depends heavily on the constructs of partial derivatives.

Economics

Partial derivatives play a prominent role in economics, in which most functions describing economic behavior posit that the behavior depends on more than one variable. For example a societal consumption function may describe the amount spent on consumer goods as depending on both income and wealth; the marginal propensity to consume is then the partial derivative of the consumption function with respect to income.

CONCLUSION

By the definition of partial derivates we are finding first order and second order up to higher order derivates and their functions of differentials and gradient of a function and their function of chain rules applying on the functions. Finding function of tangent approximations and their function is continuous and differentiable and mainly partial derivates are applying in economics and their function behavior.

REFERENCES

1) Hansen PC. Analysis of discrete ill-posed problems by means of the L-curve. SIAM Rev. 1992;34:561–580.

2) Trappeniers NJ, Wassenaar T, Abels JC. Isotherms and thermodynamic properties of methane at temperatures between 0C and 150C and at densities up to 570 Amagat. Physica. 1979;98A:289– 297

3) Setzmann U, Wagner W. A new equation of state and tables of thermodynamic properties for methane covering the range from the melting line to 625 K at pressures up to 1000 MPa. J Phys Chem Ref Data. 1991;20:1061–1155.

4) Sandler SI. Chemical Biochemical and Engineering Thermodynamics, 4th ed. New York, NY: Wiley, 2006.

5) Yeow YL, Ge J, Leong Y-K, Khan A. Investigating the properties of aqueous monoisopropanolamine using density data from 283.15K to 353.15K. Int J Thermophys. 2009;30:448–463.

6) Wahba G. Spline Models for Observational Data. Philadelphia, PA: SIAM, 1990.