

DOUBLE INTEGRALS AND THEIR EVALUATION BY ITERATED INTEGRALS AND THEIR APPLICATIONS

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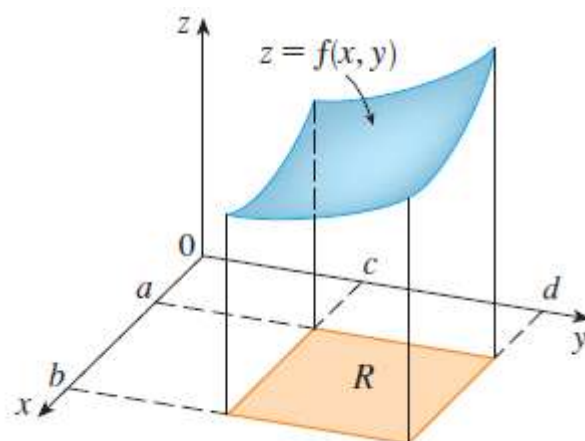
Definition: Let R be a bounded region in the xy plane and f a function continuous on R .

a. The a double integral of f over the rectangle R is

$\iint_R f(x,y) dA = \lim_{m,n \rightarrow \infty} \sum_{j=1}^m \sum_{i=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$ if this limit exists, where m is subintervals of R on x – axis and n is subintervals of R on y – axis.

(x_{ij}^*, y_{ij}^*) a sample point of R_{ij} of part of R .

b. If f is non negative and integrable on R , then the volume V of the solid region between the graph of f and R is given by $V = \iint_R f(x,y) dA$



Suppose that f is a function of two variables that is integrable on the

rectangle $[a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y) dy$ to mean that x held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called **partial integration** with respect to y . Now $\int_c^d f(x, y) dy$ is a number that depends on the value of x , so it defines a function of x .

$$A(x) = \int_c^d f(x, y) dy$$

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b A(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx \cdots (1)$$

The integral on the right side of equation 1 is called an iterated integral.

Usually the brackets are omitted. Thus

$$\int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx \cdots (2)$$

Means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly the iterated integral

$$\int_c^d \int_a^b f(x, y) dx dy = \int_c^d \left[\int_a^b f(x, y) dx \right] dy \cdots (3)$$

means that we first integrate with respect to x (holding y fixed) from x = a to x = b and then we integrate the resulting function of y with respect to y from y = c to y = d. Notice that in both equations 2 and 3 we work from the inside out.

Keywords: Definition of Double Integral, Polar Coordinates, Applications.

Example 1. Evaluate the iterated integrals,

$$(a) \int_0^3 \int_1^2 x^2 y dy dx \quad (b) \int_1^2 \int_0^3 x^2 y dx dy$$

Solution: (a) Regarding x as constant, we obtain

$$\int_1^2 x^2 y dy = \left[\frac{x^2 y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \frac{3}{2} x^2$$

Thus the function A in the preceding discussion is given by $A(x) = \frac{3}{2} x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y dy dx = \int_0^3 \left[\int_1^2 x^2 y dy \right] dx = \int_0^3 \frac{3}{2} x^2 dx = \left(\frac{x^3}{2} \right) \Big|_0^3 = \frac{27}{2}$$

(b) Here we first integrate with respect to x:

$$\begin{aligned} \int_1^2 \int_0^3 x^2 y dx dy &= \int_1^2 \left[\int_0^3 x^2 y dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy \\ &= \int_1^2 9y dy = \left(\frac{9y^2}{2} \right) \Big|_1^2 = \frac{27}{2} \end{aligned}$$

Theorem(Fubini's Theorem): If f is continuous on the rectangle

$$R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}, \text{ then}$$

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

More generally, this is true if we assume that f is bounded on R , f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 2: Evaluate the double integral $\iint_R (x - 3y^2) dA$, where

$$R = \{(x, y) | 0 \leq x \leq 2, 1 \leq y \leq 2\}.$$

Solution: Fubini's Theorem gives

$$\begin{aligned} \iint_R (x - 3y^2) dA &= \int_0^2 \int_1^2 (x - 3y^2) dy dx = \int_0^2 [xy - y^3]_{y=1}^{y=2} dx \\ &= \int_0^2 (x - 7) dx = \left(\frac{x^2}{2} - 7x \right) \Big|_0^2 = -12 \end{aligned}$$

Example 3: Evaluate $\iint_R y \sin(x, y) dA$, where $R = [1, 2] \times [0, \pi]$.

Solution 1: If we first integrate with respect to x , we get

$$\begin{aligned} \iint_R y \sin(x, y) dA &= \int_0^\pi \int_1^2 y \sin(x, y) dx dy = \int_0^\pi [-\cos(x, y)]_{x=1}^{x=2} dy \\ &= \int_0^\pi (-\cos 2y + \cos y) dy \\ &= \left(-\frac{1}{2} \sin 2y + \sin y \right) \Big|_0^\pi = 0 \end{aligned}$$

Solution 2: If we reverse the order of the integration, we get

$$\iint_R y \sin(x, y) dA = \int_1^2 \int_0^\pi y \sin(x, y) dy dx$$

To evaluate the inner integral, we use integration by parts with

$$\begin{aligned} u &= y & dv &= \sin(xy) dy \\ du &= dy & v &= \frac{-\cos(xy)}{x} \end{aligned}$$

$$\begin{aligned} \text{and so } \int_0^\pi y \sin(xy) dy &= \frac{-y \cos(xy)}{x} \Big|_{y=0}^{y=\pi} + \frac{1}{x} \int_0^\pi \cos(xy) dy \\ &= \frac{-\pi \cos \pi x}{x} + \frac{1}{x^2} [\sin(xy)]_{y=0}^{\pi} \\ &= \frac{-\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2} \end{aligned}$$

If we now integrate the first term by parts with $u = \frac{-1}{x}$ and $dv = \pi \cos \pi x dx$, we get

$$dv = \frac{dx}{x^2}, v = \sin \pi x, \text{ and } \int \left(\frac{-\pi \cos \pi x}{x} \right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2} dx$$

$$\text{Therefore } \int \left(\frac{-\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2} \right) dx = -\frac{\sin \pi x}{x}$$

$$\text{and so } \int_1^2 \int_0^\pi y \sin(x, y) dy dx = \left[-\frac{\sin \pi x}{x} \right]_1^2 = \frac{-\sin 2\pi}{2} + \sin \pi = 0$$

Double Integral over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over

regions D of more general shape, such as the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure

2. Then we define a new function F with domain R by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \in R \text{ but not in } D \end{cases} \quad \dots (1)$$

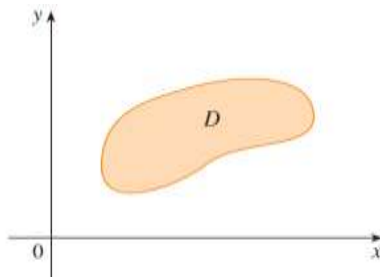


FIGURE 1

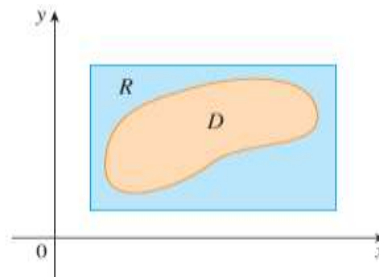


FIGURE 2

If F

is integrable over R , then we define the double integral of f over D by

$$\iint_D f(x, y) dA = \iint_R F(x, y) dA$$

where F is given by Equation 1.

Definitions:

1. A plane region D is said to be of **type I** if it lies between the graphs of two continuous function of x , that is,

$$D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \text{ where } g_1 \text{ and } g_2 \text{ are continuous on } [a, b].$$

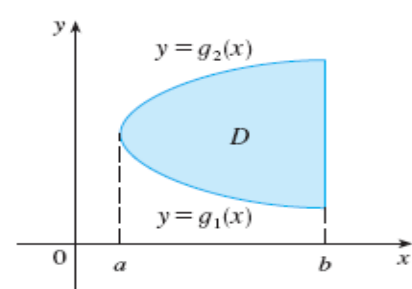
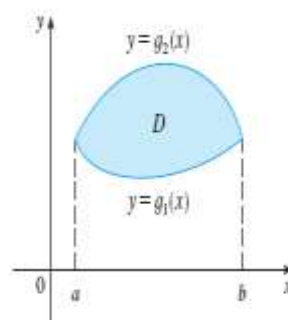
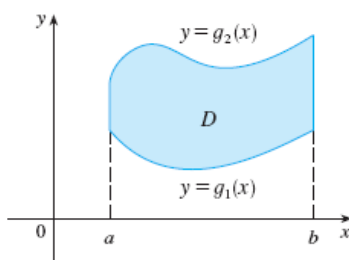


FIGURE 5 Some type I regions

2. If f is continuous on a type I region D such that

$$D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

$$\text{then } \iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

3. A plane region D is said to be of type II if it lies between the graphs of two continuous function of y , that is,

$$D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\} \text{ where } h_1 \text{ and } h_2 \text{ are continuous on } [c, d].$$

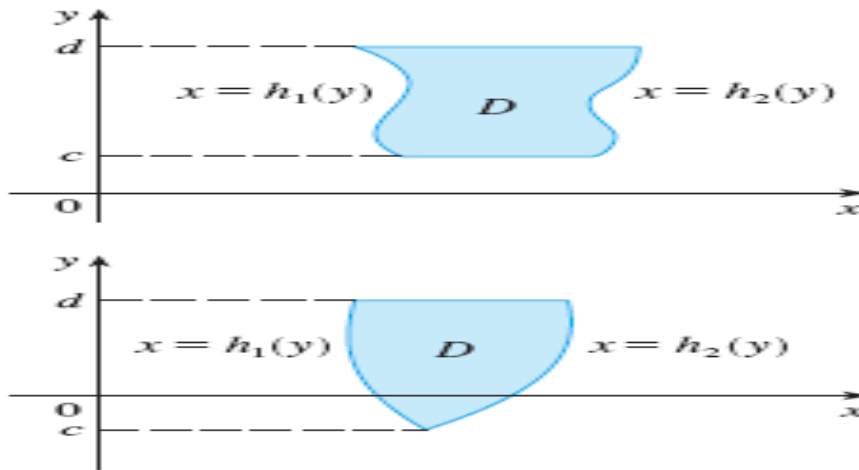


FIGURE 7
Some type II regions

$$\text{So } \iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Example 6: Evaluate $\iint_D (x + 2y) dA$, where D is the region bounded by the parabolas

$$y = 2x^2 \text{ and } y = 1 + x^2.$$

Solution: The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D , is a type I region but not a type II region and we can write

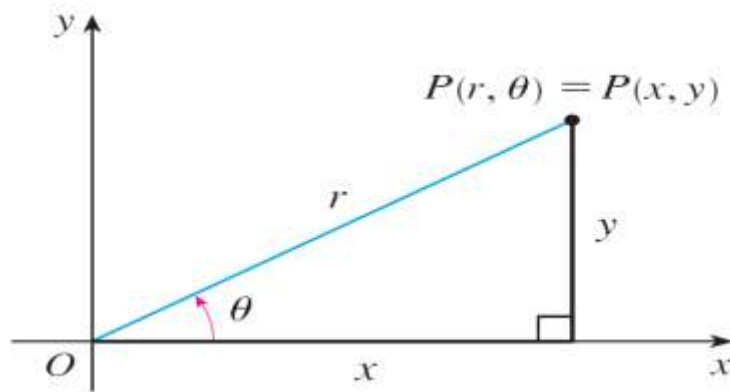
$$D = \{(x, y) | -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$,

$$\begin{aligned} \iint_D (x + 2y) dA &= \int_{-1}^1 \int_{2x^2}^{1+x^2} (x + 2y) dy dx = \int_{-1}^1 [xy + y^2]_{y=2x^2}^{y=1+x^2} dx \\ &= \int_{-1}^1 [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2] dx \\ &= \int_{-1}^1 (-3x^4 - x^3 + 2x^2 + x + 1) dx \\ &= \left(\frac{-3x^5}{5} - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x \right) \Big|_{-1}^1 = \frac{32}{15} \end{aligned}$$

Double Integrals in polar coordinates

Change to polar coordinates in a double integral

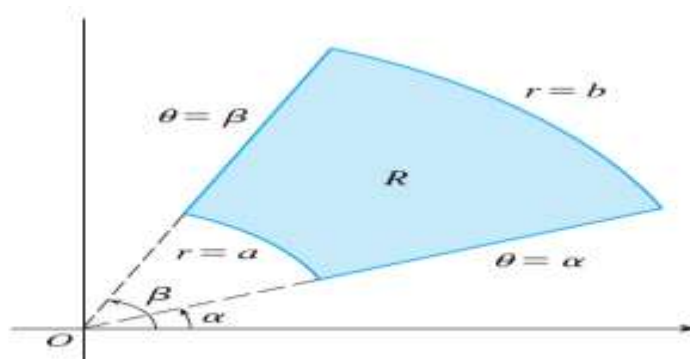


From the above figure we can see that the polar coordinate (r, θ) of a point are related to the rectangular coordinate (x, y) by the equations: $x^2 + y^2 = r^2$

$$x = r \cos \theta, y = r \sin \theta$$

If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$



Example 1: Evaluate $\iint_R (3x + 4y^2) dA$, where R is the region in the upper half – plane

bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

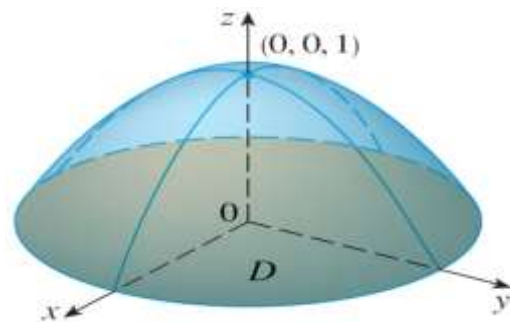
Solution: The region R can be described as

$$R = \{(x, y) | y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$$

In polar coordinates it is given by $1 \leq r \leq 2, 0 \leq \theta \leq \pi$. Therefore,

$$\begin{aligned}
 \iint_R (3x + 4y^2) dA &= \int_0^\pi \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) r dr d\theta \\
 &= \int_0^\pi \int_1^2 (3r^2 \cos \theta + 4r^3 \sin^2 \theta) dr d\theta \\
 &= \int_0^\pi [r^2 \cos \theta + r^4 \sin^2 \theta]_{r=1}^{r=2} d\theta = \int_0^\pi (7 \cos \theta + 15 \sin^2 \theta) d\theta \\
 &= \int_0^\pi \left[7 \cos \theta + \frac{15}{2} (1 - \cos 2\theta) \right] d\theta \\
 &= 7 \sin \theta + \frac{15\theta}{2} - \frac{15}{4} \sin 2\theta \Big|_0^\pi = \frac{15\pi}{2}
 \end{aligned}$$

Example 2: Find the volume of the solid bounded by the plane $z = 0$ and the Paraboloid $z = 1 - x^2 - y^2$.



Solution: If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leq 1$. In polar coordinates D is given by $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$\begin{aligned}
 V &= \iint_D (1 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta \\
 &= \int_0^{2\pi} d\theta \int_0^1 (r - r^3) dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2}.
 \end{aligned}$$

Example 3: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy -plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution: The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square, $(x - 1)^2 + y^2 = 1$

In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r \cos \theta$, so the boundary circle becomes $r^2 = 2r \cos \theta$, or $r = 2 \cos \theta$. Thus the disk D is given by

$$D = \left\{ (r, \theta) : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2 \cos \theta \right\} \text{ and we, have}$$

$$\begin{aligned} V &= \iint_D (x^2 + y^2) dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 r dr d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_0^{2\cos\theta} d\theta \\ &= 4 \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta = 8 \int_0^{\pi/2} \cos^4 \theta d\theta = 8 \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta \\ &= 2 \int_0^{\pi/2} \left[1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta) \right] d\theta \\ &= 2 \left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta \right]_0^{\pi/2} = 2 \left(\frac{3}{2} \right) \left(\frac{\pi}{2} \right) = \frac{3\pi}{2} \end{aligned}$$

Applications of Double Integrals

Area, center of mass of plane region, surface area

If f is continuous on a polar region of the form

$$D = \{(r, \theta) = \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

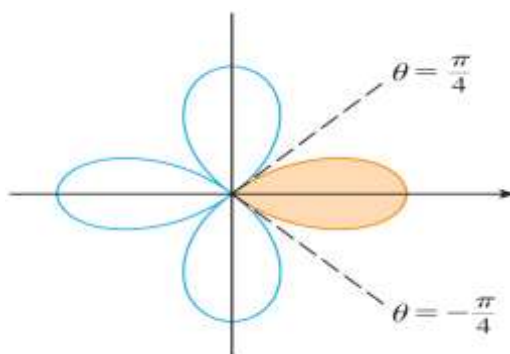
$$\text{then } V = \iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

In particular, taking $f(x, y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$ in the formula, we see that the area of the region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is:

$$\begin{aligned} A(D) &= \iint_D 1 dA = \int_{\alpha}^{\beta} \int_0^{h(\theta)} r dr d\theta = \int_{\alpha}^{\beta} \left[\frac{r^2}{2} \right]_0^{h(\theta)} d\theta \\ &= \int_{\alpha}^{\beta} \frac{1}{2} [h(\theta)]^2 d\theta \end{aligned}$$

Example 1: Use a double integral to find the area enclosed by one loop of the four leaved rose $r = \cos 2\theta$.

Solution: From the sketch of the curve in figure below, we see that a loop is given by the region $D = \{(r, \theta) = -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq \cos^2 \theta\}$ so the area is



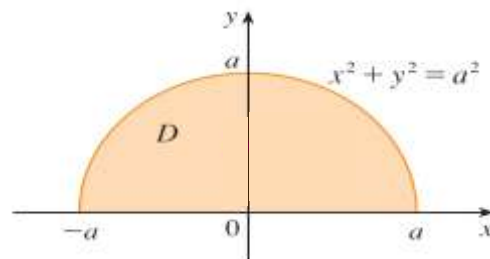
Center of Mass

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a plane occupying the region D and having

density function $\rho(x, y)$ are $\bar{x} = \frac{1}{m} \iint_D x \rho(x, y) dA$ $\bar{y} = \frac{1}{m} \iint_D y \rho(x, y) dA$

Where the mass m is given by $m = \iint_D \rho(x, y) dA$

Example 2: The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass the lamina.



Solution: Let's place the lamina as the upper half of the circle $x^2 + y^2 = a^2$. Then the distance from a point (x, y) to the center of the circle (the origin) is $\sqrt{x^2 + y^2}$. Therefore the density function is $\rho(x, y) = K\sqrt{x^2 + y^2}$ where K is some constant. Both the density function and the shape of the lamina suggest that we convert to polar coordinates. Then $\sqrt{x^2 + y^2} = r$ and the region D is given by $0 \leq r \leq a, 0 \leq \theta \leq \pi$. Thus the mass of the lamina is

$$\begin{aligned} m &= \iint_D \rho(x, y) dA = \iint_D K\sqrt{x^2 + y^2} dA = \int_0^\pi \int_0^a (kr) r dr d\theta \\ &= K \int_0^\pi d\theta \int_0^a r^2 dr = K \pi \frac{r^3}{3} \Big|_0^a = \frac{K\pi a^3}{3} \end{aligned}$$

Both the lamina and the density function are symmetric with respect to the y – axis, so the center of mass must lie on the y – axis, that is, $\bar{x} = 0$. The y – coordinate is given by

$$\begin{aligned} \bar{y} &= \frac{1}{m} \iint_D y \rho(x, y) dA = \frac{3}{K\pi a^3} \int_0^\pi \int_0^a r \sin \theta (kr) r dr d\theta \\ &= \frac{3}{\pi a^3} \int_0^\pi \sin \theta d\theta \int_0^a r^3 dr = \frac{3}{\pi a^3} [-\cos \theta]_0^\pi \left[\frac{r^4}{4} \right]_0^a \\ &= \frac{3}{\pi a^3} \frac{2a^4}{4} = \frac{3a}{2\pi}. \end{aligned}$$

Therefore the center of mass is located at the point $\left(0, \frac{3a}{2\pi}\right)$.

Surface Area

Definition: Let R be a region, and let f have continuous partial derivatives on R . If Σ is the graph of f on R , then the surface area S of Σ is defined by

$$S = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA$$

Example 3: Let R be the rectangular region bounded by the lines

$$x = 0, x = 3, y = 0, y = 2$$

and let $f(x, y) = \frac{2}{3}x^{3/2}$.

Find the surface area S of the portion of the graph of f that lies over R .

Solution: Notice that $f_x(x, y) = x^{1/2}$ and $f_y(x, y) = 0$ for (x, y) in R consequently

$$\begin{aligned} S &= \iint_R \sqrt{\left(x^{1/2}\right)^2 + 0 + 1} dA = \int_0^3 \int_0^2 \sqrt{x+1} dy dx \\ &= 2 \int_0^3 \sqrt{x+1} dx = \left(\frac{4}{3}(x+1)^{3/2}\right)\bigg|_0^3 = \frac{28}{3} \end{aligned}$$

CONCLUSION

By the help of double integrals we are finding the volume of the integrals and given double integral is converted to polar coordinates and finally we finding the applications of double integrals like Area, center of mass of plane region, surface area, centre of mass.

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