OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS AND FIXED POINTS IN 2 NON-ARCHIMEDEAN MENGER PM SPACE

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ABSTRACT

The concept of occasionally weakly compatible mappings is used to prove a common fixed point theorem. The theorem thus obtained is a generalization and extension of the result of Khan and Sumitra [13] in a 2 non-Archimedean Menger PM-space.

Keywords:* *2 Non-Archimedean Menger Probabilistic Metric Space, Common Fixed Points, Compatible Maps, Occasionally Weakly Compatible Maps.

AMS Subject Classification:* *Primary 47H10, Secondary 54H25.

I INTRODUCTION

The notion of non-Archimedean Menger space has been established by Istrătescu and Crivat [9]. The existence of fixed point of mappings on non-Archimedean Menger space has been given by Istrătescu [6]. This has been the extension of the results of Sehgal and Bharucha - Reid [16] and Sherwood [17] on a Menger space. Cho et. al. [3] proved a common fixed point theorem for compatible mappings in non-Archimedean Menger PM-space. Achari [1] studied the fixed points of quasi-contraction type mappings in non-Archimedean PM-spaces and generalized the results of Istrătescu [7]. Recently Khan and Sumitra [13] proved a common fixed point theorem for three pointwise R-weakly commuting mappings in complete non-Archimedean Menger PM-spaces. In the present paper we prove a unique common fixed point theorem for four occasionally weakly compatible self maps in non-Archimedean Menger PM-spaces without using the notion of continuity. Our result generalizes and extends the results of Khan and Sumitra [13] and others.

II PRELIMINARIES

Definition 2.1. [20] Let X be a non-empty set and $\mathcal{D}$ be the set of all left-continuous distribution functions. An ordered pair (X, $\mathcal{F}$) is called a 2 non-Archimedean probabilistic metric space (briefly, a N.A. PM-space) if $\mathcal{F}$ is a mapping from X×X×X into $\mathcal{D}$ satisfying the following conditions (the distribution function $\mathcal{F}$(u,v,w) is denoted by $F_{u,v,w}$ for all u,v, w $\in$ X) :

(PM-1) F(u, v, w; t) = 1, for all t > 0, if and only at least two of the three points are equal;

(PM-2) $F(u,v,w; t) = F(u,w,v; t) = F(w,v,u; t)$;

(PM-3) $F(u, v, w; 0) = 0$;

(PM-4) If $F(u, v, s; t_1) = F(u, s, w; t_2) = 1$ and $F(s, v, w; t_3) = 1$ then $F(u, v, w; \max\{t_1, t_2, t_3\}) = 1$,

for all $u, v, w, s \in X$ and $x_1, x_2, x_3 \geq 0$.

Definition 2.2. [20] A t-norm is a function $\Delta : [0,1] \times [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, nondecreasing in each coordinate and $\Delta(a,1, 1) = a$ for every $a \in [0,1]$.

Definition 2.3. [20] A 2 *N.A. Menger PM-space* is an ordered triple $(X, \mathcal{F}, \Delta)$, where $(X, \mathcal{F})$ is a 2 non-Archimedean PM-space and Δ is a t-norm satisfying the following condition:

(PM-5) $F_{u, v, w}(\max\{x, y, z\}) \geq \Delta(F_{u,v,s}(x_1), F_{u,s,w}(x_2), F_{s,v,w}(x_3))$,

for all $u, v, w, s \in X$ and $x_1, x_2, x_3 \geq 0$.

Definition 2.4. [20] A 2 N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that $g(F_{x,y, z}(t)) \leq g(F_{x, y, a}(t)) + g(F_{x, a, z}(t)) + g(F_{a, y, z}(t))$

for all $x, y, z, a \in X$ and $t \geq 0$, where $\Omega = \{g \mid g : [0,1] \to [0, \infty)$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty\}$.

Definition 2.5. [20] A 2 N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is said to be of type $(D)_g$ if there exists a $g \in \Omega$ such that $g(\Delta(t_1, t_2, t_3)) \leq g(t_1) + g(t_2) + g(t_3)$ for all $t_1, t_2, t_3 \in [0,1]$.

Remark 2.1. [20]

(1) If a 2 N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is of type $(D)_g$ then $(X, \mathcal{F}, \Delta)$ is of type $(C)_g$.

(2) If a 2 N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ is of type $(D)_g$, then it is metrizable, where the metric d on X is defined by

$d(x,y) = \int_0^1 g(F_{x,y,a}(t))d(t)$ for all $x, y, a \in X$. (*)

Throughout this paper, suppose $(X, \mathcal{F}, \Delta)$ is a complete 2 N.A. Menger PM-space of type $(D)_g$ with a continuous strictly increasing t-norm Δ.

Let $\phi : [0, +\infty) \to [0, +\infty)$ be a function satisfied the condition (Φ) :

(Φ) ϕ is upper-semicontinuous from the right and $\phi(t) < t$ for all $t > 0$.

Lemma 2.1. [20] If a function $\phi : [0,+ \infty) \to [0,+\infty)$ satisfies the condition (Φ), then we have

(1) For all $t \geq 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is n^{th} iteration of $\phi(t)$.

(2) If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$, $n = 1, 2, \ldots$ then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$ for all $t \geq 0$, then $t = 0$.

Definition 2.6. [20] Let $A, S : X \to X$ be mappings. A and S are said to be compatible if $\lim_{n\to\infty} g(F(ASx_n,SAx_n,a; t)) = 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z in X.

Definition 2.7. [20] Self maps A and S of a 2 N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if Ap = Sp for some $p \in X$ then ASp = SAp.

Definition 2.8. Self maps A and S of a 2 N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Lemma 2.2. [3] Let $A, B, S, T : X \to X$ be mappings satisfying the condition (1) and (2) as follows :

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

(2) $g(F(Ax,By; t)) \leq \phi(\max\{g(F(Sx,Ty; t)), g(F(Sx, Ax; t)), g(F(Ty, By; t)),$

$$\tfrac{1}{2}(g(F(Sx, By; t)) + g(F(Ty, Ax; t)))\})$$

for all t > 0, where a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ). Then the sequence $\{y_n\}$ in X, defined by $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for n = 0, 1, 2, ..., such that

$$\lim_{n \to \infty} g(F(y_n,y_{n+1}; t)) = 0 \quad \text{for all } t > 0$$ is a Cauchy sequence in X.

III MAIN RESULT

Theorem 3.1. Let (X, F, Δ) be a complete 2 N.A. Menger PM-space and $A, B, S, T : X \to X$ be mappings satisfying the conditions

(3.1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;

(3.2) the pairs (A, S) and (B, T) are occasionally weakly compatible and

(3.3) $g(F(Ax, By, a; t)) \leq \phi[\max\{g(F(Sx, Ty, a; t)), g(F(Sx, Ax, a; t)), g(F(Ty, By, a; t)), \tfrac{1}{2}(g(F(Sx, By, a; t)) + g(F(Ty, Ax, a; t)))\}]$ for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A, B, S and T have a unique common fixed point in X.

Proof. Since $A(X) \subseteq T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for this x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_n = Ax_{2n} = Tx_{2n+1},\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 1, 2, \ldots . \tag{1}$$

Let $M_n = g(F(Ax_n, Bx_{n+1}, a; t) = g(F(y_n, y_{n+1}, a; t)$ for n = 1, 2, … . Then

$$M_{2n} = g(F(Ax_{2n}, Bx_{2n+1}, a; t)$$

$$\leq \phi[\max\{g(F(Sx_{2n}, Tx_{2n+1}, a; t)), g(F(Sx_{2n}, Ax_{2n}, a; t)), g(F(Tx_{2n+1}, Bx_{2n+1}, a; t)),$$

$$\tfrac{1}{2}(g(F(Sx_{2n}, Bx_{2n+1}, a; t)) + g(F(Tx_{2n+1}, Ax_{2n}, a; t)))\}]$$

$\leq \phi[\max\{g(F(y_{2n-1}, y_{2n}, a; t)), g(F(y_{2n-1}, y_{2n}, a; t)), g(F(y_{2n}, y_{2n+1}, a; t)),$

$\frac{1}{2}(g(F(y_{2n-1}, y_{2n+1}, a; t)) + g(F(y_{2n}, y_{2n}, a; t)))\}]$

i.e. $M_{2n} \leq \phi[\max\{M_{2n-1}, M_{2n-1}, M_{2n}, \frac{1}{2}(M_{2n-1} + M_{2n})\}]$. (2)

If $M_{2n} > M_{2n-1}$ then by (2),

$M_{2n} \geq \phi(M_{2n})$, a contradiction.

If $M_{2n-1} > M_{2n}$ then by (2),

$M_{2n} \leq \phi(M_{2n-1})$.

So by Lemma 2.1, we have $\lim_{n\to\infty} M_{2n} = 0$, i.e.

$\lim_{n\to\infty} g(F(Ax_{2n}, Bx_{2n+1}, a; t)) = 0$

i.e. $\lim_{n\to\infty} g(F(y_{2n}, y_{2n+1}, a; t)) = 0$.

Similarly, we can show that

$\lim_{n\to\infty} g(F(Bx_{2n+1}, Ax_{2n+2}, a; t)) = 0$

i.e. $\lim_{n\to\infty} g(F(y_{2n+1}, y_{2n+2}, a; t)) = 0$.

Thus, we have

$\lim_{n\to\infty} g(F(Ax_{2n}, Bx_{n+1}, a; t)) = 0$ for all $t > 0$

i.e. $\lim_{n\to\infty} g(F(y_n, y_{n+1}, a; t)) = 0$ for all $t > 0$. (3)

Hence, by Lemma 2.2, the sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete, so the sequence $\{x_n\}$ converges to a point z in X and so the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ also converges to the limit z.

Since $B(X) \subseteq S(X)$, there exists a point $u \in X$ such that $z = Su$. Then, using (3.3), we have

$g(F(Au, z, a; t)) \leq g(F(Au, Bx_{2n-1}, a; t)) + g(F(Bx_{2n-1}, z, a; t))$

$\leq \phi[\max\{g(F(Su, Tx_{2n-1}, a; t)), g(F(Su, Au, a; t)), g(F(Tx_{2n-1}, Bx_{2n-1}, a; t),$

$\frac{1}{2}(g(F(Su, Bx_{2n-1}, a; t)) + g(F(Tx_{2n-1}, Au, a; t)))\}]$.

Letting $n \to \infty$, we get

$g(F(Au, z, a; t)) \leq \phi[\max\{g(z, z, a; t)), g(F(z, Au, a; t)), g(F(z, z, a; t)), \frac{1}{2}(g(F(z, z, a; t)) + g(F(z, Au, a; t)))\}]$

$= \phi[\max\{0, g(F(z, Au, a; t)), 0, \frac{1}{2}(0 + g(F(z, Au, a; t)))\}]$

$\leq \phi(g(F(Au, z, a; t)))$

for all t > 0, which implies that g(F(Au, z, a; t)) = 0 for all t > 0 by Lemma 2.1. Therefore, Au = Su = z. Since A(X) $\subseteq$ T(X), there exists a point v in X such that z = Tv. Again, using (3.3), we have

$$\begin{aligned} g(F(z, Bv, a; t)) &= g(F(Au, Bv, a; t)) \\ &\le \phi[\max\{g(F(Su, Tv, a; t)), g(F(Su, Au, a; t)), g(F(Tv, Bv, a; t)), \\ &\qquad ½(g(F(Su, Bv, a ; t)) + g(F(Tv, Au, a; t)))\}] \\ &\le \phi[\max\{g(F(z, z, a; t)), g(F(z, z, a; t)), g(F(z, Bv, a; t)), \\ &\qquad ½(g(F(z, Bv ; t)) + g(F(z, z; t)))\}] \\ &= \phi[\max\{0, 0, g(F(z, Bv, a; t)), ½(g(F(z, Bv, a ; t)) + 0)\}] \\ &\le \phi(g(F(Bv, z, a; t))) \quad \text{for all } t > 0, \end{aligned}$$

which implies that g(F(Bv, z, a; t)) = 0 for all t > 0 by Lemma 2.1.

Therefore, Bv = Tv = z. Since A and S are occasionally weakly compatible mappings, ASz = SAz i.e. Az = Sz.

Now we show that z is a fixed point of A. If Az $\neq$ z, then by (3.3), we have

$$\begin{aligned} g(F(Az, z, a; t)) &= g(F(Az, Bv, a; t)) \\ &\le \phi[\max\{g(F(Sz, Tv, a; t)), g(F(Sz, Az, a; t)), g(F(Tv, Bv, a; t)), \\ &\qquad ½(g(F(Sz, Bv, a ; t)) + g(F(Tv, Az, a; t)))\}] \\ &\le \phi[\max\{g(F(Az, z, a; t)), 0, 0, ½(g(F(Az, z, a; t)) + g(F(z, Az, a; t)))\}] \\ &\le \phi(g(F(Az, z, a; t))) \quad \text{for all } t > 0, \end{aligned}$$

which implies that g(F(Az, z, a; t) = 0 for all t > 0 by Lemma 2.1. Therefore, Az = z.

Hence, Az = Sz = z.

Similarly, as B and T are occasionally weakly compatible mappings, we have

Bz = Tz = z, since by (3.3), we have

$$\begin{aligned} g(F(z, Bz, a; t)) &= g(F(Az, Bz, a; t)) \\ &\le \phi[\max\{g(F(Sz, Tz, a; t)), g(F(Sz, Az, a; t)), g(F(Tz, Bz, a; t)), \\ &\qquad ½(g(F(Sz, Bz, a ; t)) + g(F(Tz, Az, a; t)))\}] \\ &\le \phi[\max\{g(F(z, Bz, a; t)), 0, 0, ½(g(F(z, Bz, a; t)) + g(F(Bz, z, a; t)))\}] \\ &\le \phi(g(F(Bz, z, a; t))) \quad \text{for all } t > 0, \end{aligned}$$

which implies that g(F(Bz, z, a; t) = 0 for all t > 0 by Lemma 2.1. Therefore, Bz = z.

Hence, Bz = Tz = z.

Thus, Az = Bz = Sz = Tz = z, that is, z is a common fixed point of A, B, S and T.

Finally, in order to prove the uniqueness of z, suppose that w is another common fixed point of A, B, S and T. Then by (3.3), we have

$$g(F(z, w, a; t)) = g(F(Az, Bw, a; t))$$
$$\leq \phi[\max\{g(F(Sz, Tw, a; t)), g(F(Sz, Az, a; t)), g(F(Tw, Bw, a; t)), \tfrac{1}{2}(g(F(Sz, Bw, a; t)) + g(F(Tz, Aw, a; t)))\}]$$
$$\leq \phi(g(F(z, w, a; t)) \text{ for all } t > 0,$$

which implies that g(F(z, w, a; t)) = 0 for all t > 0 by Lemma 2.1.

Hence, z = w.

Therefore, z is a unique common fixed point of A, B, S and T.

Corollary 3.1. Let A, S, T : X → X be the mappings satisfying

(i) $A(X) \subseteq S(X) \cap T(X)$,

(ii) the pairs {A, S} and {A, T} are occasionally weakly compatible and

(iii) $g(F(Ax, Ay, a; t)) \leq \phi[\max\{g(F(Sx, Ty, a; t)), g(F(Sx, Ax, a; t)), g(F(Ty, Ay, a; t)), \tfrac{1}{2}(g(F(Sx, Ay, a; t)) + g(F(Ty, Ax, a; t)))\}]$, for every x, y ∈ X, where ϕ satisfies the condition (Φ). Then A, S and T have a unique common fixed point in X.

Corollary 3.2. Let A, S : X → X be the mappings satisfying

(i) $A(X) \subseteq S(X)$,

(ii) the pairs {A, S} is occasionally weakly compatible and

(iii) $g(F(Ax, Ay, a; t)) \leq \phi[\max\{g(F(Sx, Sy, a; t)), g(F(Sx, Ax, a; t)), g(F(Sy, Ay, a; t)), \tfrac{1}{2}(g(F(Sx, Ay, a; t)) + g(F(Sy, Ax, a; t)))\}]$, for every x, y ∈ X, where ϕ satisfies the condition (Φ). Then A and S have a unique common fixed point in X.

Remark 3.1. In Theorem 3.1, if S and T are continuous and pairs {A, S} and {B, T} are compatible instead of condition (3.2), the theorem remains true.

Remark 3.2. In our generalization the inequality condition (3.3) satisfied by the mappings A, B, S and T 2 non-Archimedean Menger PM-space is stronger than that of Theorem 2 of Khan and Sumitra [13] and Theorem 1.9 of Vasuki [21] in non-Archimedean Menger PM-space.

IV ACKNOWLEDGEMENT

Authors are thankful to **Dr. Arihant Jain** for the fruitful comments on this research work.

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