GENERAL EQUATION FOR BINDING ENERGY

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ABSTRACT

The purpose of this technical paper is to put forth a general equation for binding energy; an equation which is applicable to every object in the universe which performs simultaneous rotating and revolving motion. We apply the equation to the massive moon, satellites and even the tiniest of particles like electrons and will discuss ionization energy which is basically binding energy for electrons and calculate the values for some common elements. The paper is focused towards deriving the equation of binding energy applicable to objects performing an elliptical motion, such as comets/meteors/asteroids. The paper also mentions the significance of binding energy with a reference to the Chelyabinsk meteor. By deriving a general equation we understand how different objects placed in different parts of the universe, be it as huge as planets or as tiny as electrons, all (rotating and revolving) objects behave in a similar manner and are bound by binding energy which can be obtained by a single formula; hence, shedding light on the laws of physics and the grand design of the universe.

Keywords : Binding Energy, Elliptical Motion, General Equation, Halley's Comet, Chelyabinsk Meteor, Ionization Energy.

I INTRODUCTION

Binding energy in simple terms is the energy required to keep and maintain an object within the orbit of an entity to which it is bound. In other words, it can also be considered as the amount of energy required to permanently remove the orbiting object from the influence of attraction. There are various forms of attraction such as Gravitational forces or Coulombic forces of attraction in case of charged objects.

The general equation for binding energy is proposed so as to facilitate the calculation of binding energy. The usefulness and practicality of the equation is demonstrated by applying it to some well-known entities such as the International Space Station, Moon, and Halley's Comet etc. most of which are objects that revolve in an elliptical orbit. In short, we discuss an easy to remember equation which has innumerable applications and may prove to be highly useful to students, engineers and other academicians.

II PRE-REQUISITES

In order to swiftly understand all the equations and their respective derivations, it is important to know the following:

$$a_l = r\alpha$$

$$a_c = r\omega^2$$

$$E = \frac{1}{2} I\omega^2$$

$$\tau = I\alpha$$

Where,

$a_l = Linear\ acceleration$

$a_c = Centripetal\ acceleration$

$r = Radius\ of\ orbit$

$\alpha = Angular\ acceleration$

$\omega = Angular\ velocity$

$E = Energy$

$I = Moment\ of\ Inertia$

$\tau = Torque$

III DERIVATION OF FORMULA

Before deriving the formula we need to understand the different kinds of orbital motion:

3.1 Types of orbital motion

There are broadly 3 different kinds of orbital motion:

3.1.1 Uniform Closed Orbit

These kinds of orbits are closed. They may not necessarily be circular. They may also be elliptical in nature. These curves occur when the centripetal acceleration is equal to the linear acceleration.

$$\therefore a_c = a_l$$

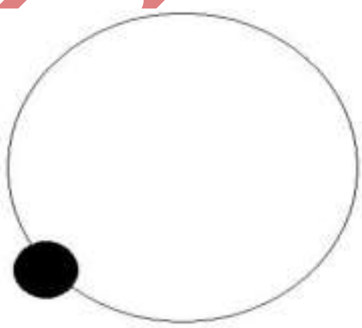

Fig.1. Uniform Closed Orbit.

3.1.2 Helically Inward Orbit

These kinds of orbits are not closed. The orbiting body tends to move inward. That is, the body gradually gets closer and closer to the centre of attraction. This results in a helically inward curve. These curves occur when the centripetal acceleration is greater than the linear acceleration.

$$\therefore\ a_c > a_l$$

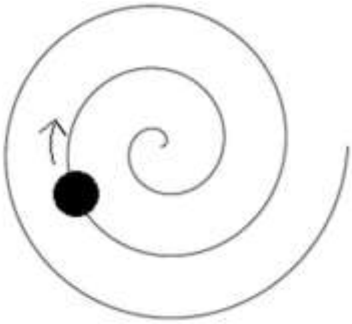

Fig.2. Helically Inward Orbit

3.1.3 Helically Outward Orbit

These kinds of orbits are also open. The orbiting body tends to move outward. That is, the body gradually gets farther away from the center of attraction. This results in a helically outward curve. These curves occur when the linear acceleration is greater than the centripetal acceleration.

$$\therefore\ a_l > a_c$$

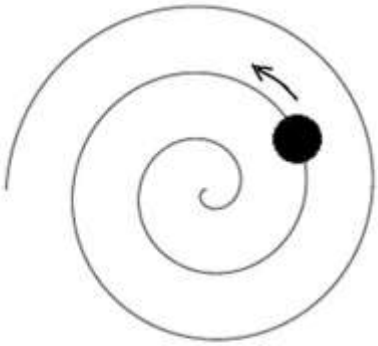

Fig.3. Helically outward orbit

3.2 Assumption

In order to obtain the formula, we make the following assumption:

For a stable and continuous revolution, the centripetal acceleration must be equal to the linear acceleration.

$$\therefore\ a_c = a_l$$

$$r\,\omega^2 = r\,\alpha$$

$$\therefore \omega^2 = \alpha$$

$$\Rightarrow E = \frac{1}{2} I\omega^2 = \frac{1}{2}\ I\alpha$$

But,

$$\because I\alpha = \tau$$

$$\therefore \boxed{E = \frac{1}{2}\tau}$$

This is the general formula for Binding Energy.

Application

3.3 The Moon

The Moon is our natural satellite and it continuously revolves around the Earth in a nearly circular orbit.

Fig.4. Moon's orbit around the Earth

$$E = \frac{1}{2}\tau$$

$$E = \frac{1}{2} F\, d$$

$$E = \frac{1}{2}\frac{G\, M_e M_m}{d^2}\, d$$

$$E = \frac{1}{2}\frac{G\, M_e M_m}{d} = \mathbf{3.810973 \times 10^{28}} \text{ N·m}$$

Where,

$M_e = Mass\ of\ Earth = 5.9736 \times 10^{24}\ kg$ [1]

$M_m = Mass\ of\ the\ Moon = 7.349 \times 10^{22}\ kg$ [2]

$d = Distance\ between\ Earth\ and\ Moon = 384{,}400\ km$

$G = Gravitational\ Constant = 6.67398 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$

Thus, the Moon requires **3.810973** × **10^ 28** N·m of energy to remain in orbit.

3.4 Satellites

Apart from the Moon, the Earth has various other artificial satellites. These satellites revolve around the Earth in a nearly circular orbit and are used for several purposes like Atmospheric study, Communications, Navigation, Weather forecasting etc.

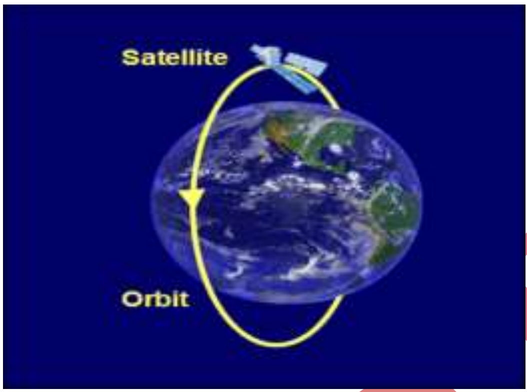

Fig.5. Satellite orbiting the earth

$$E = \frac{1}{2}\tau$$

$$E = \frac{1}{2} F d$$

$$E = \frac{1}{2} \frac{G M_E M_s}{d^2} d$$

$$E = \frac{1}{2} \frac{G M_E M_s}{d} = \mathbf{5.57 \times 10^6} M_s \text{ N·m}$$

d = Distance of geostationary orbit above Earth's surface =35,786 km

Using this value we can know the amount of energy a satellite requires to remain in orbit. Hence, we can calculate the energy which will be required to permanently remove a satellite from its orbit.

3.4.1 Binding Energy of International Space Station (ISS)

$$E = \frac{1}{2} \frac{G M_E M_{iss}}{d} = \mathbf{2.2604 \times 10^{14}} \text{ N·m}$$

Where,

d=Distance between Earth and ISS ≈370 km

M_{iss} = Mass of the ISS ≈ 419,573 kg [3]

3.5 Ionization Energy

Ionization energy is the amount of energy required to remove an electron from an atom or a molecule which is in gaseous state. If we analyze, ionization energy is also basically binding energy and so obeys the same laws and hence the same equation.

$$E = \frac{1}{2}\tau$$

$$E = \frac{1}{2} F r$$

$$E = \frac{1}{2}\frac{e Z e}{4\pi\varepsilon_0 r^2} r$$

$$E = \frac{e^2 Z}{8\pi\varepsilon_0 r} \quad ---- (1)$$

We know that for an atom,

$$r = \frac{\varepsilon_0 n^2 h^2}{\pi m_e Z e^2}$$

Substituting the value of *r* in (1) we get,

$$E = \frac{e^2 Z}{8\pi\varepsilon_0 \frac{\varepsilon_0 n^2 h^2}{\pi m_e Z e^2}}$$

$$E = \frac{e^4 Z^2 m_e}{8\varepsilon_0{}^2 n^2 h^2}$$

$$E = \frac{e^4 m_e}{8\varepsilon_0{}^2 h^2}\left(\frac{Z^2}{n^2}\right)$$

$$\therefore \boxed{E = 13.6\left(\frac{Z^2}{n^2}\right) eV}$$

Where,

E=Ionization Energy

Z=Atomic number

n=Orbit number

ε_0=Vacuum permittivity

m_e=Mass of Electron

h=Plank's constant

Table 1 Ionization Energies of some elements

Name of Element	Atomic Number	Ionization Energy (Experimental in eV)	Ionization Energy (Calculated in eV)
Hydrogen	1	13.598	13.6
Helium	2	54.417	54.4
Lithium	3	122.454	122.4
Beryllium	4	217.718	217.6
Boron	5	340.226	340
Carbon	6	489.993	489.6

Nitrogen	7	667.046	666.4
Oxygen	8	871.409	870.4
Fluorine	9	1103.117	1101.6
Neon	10	1362.199	1360

Thus, we observe that the values of Ionization energy can be accurately calculated using this formula.

Note: The slight variation in the calculated and experimental values is due to effects like electronic repulsion and shielding effect.

IV ELLIPTICAL PATH OBJECTS

4.1 Derivation of Equation for Distance between focus and point on curve

We must understand that all bodies do not perform circular motion. Sometimes a body may perform elliptical motion also. This kind of motion is seen in cases of revolution around the Sun. Planets as well as asteroids, comets etc. have an elliptical orbit around the Sun. This hence gives us the need to study binding energy in case of elliptical motion.

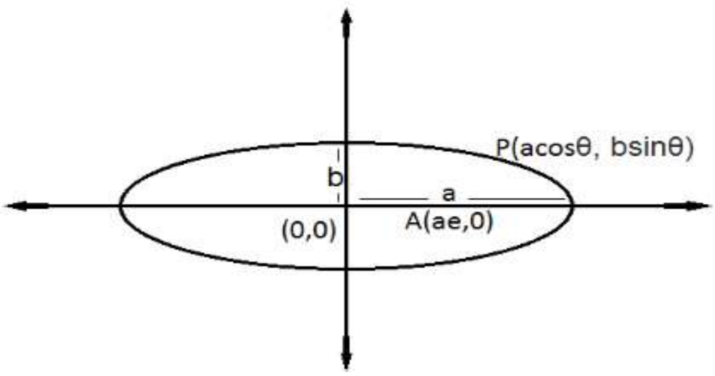

Fig.6. An ellipse with its parameters and Sun at focus

We know,

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2} \text{ - - - - - } ①$$

$$\therefore x = a\cos\theta\, ; y = b\sin\theta$$

$$\&\, x_0 = ae;\ y_0 = 0$$

Substituting the values in ①, we get,

$$r = \sqrt{(a\cos\theta - ae)^2 + (b\sin\theta - 0)^2}$$

$$r = \sqrt{a^2\cos^2\theta - 2a^2e\cos\theta + a^2e^2 + b^2\sin^2\theta} \text{ - - - - } ②$$

For an ellipse,

$$b^2 = a^2(1 - e^2)$$

Substituting value of b^2 in ②, we get,

$$r = \sqrt{a^2\cos^2\theta - 2a^2 e\cos\theta + a^2e^2 + a^2\sin^2\theta - a^2e^2\sin^2\theta}$$

$$r = \sqrt{a^2(\cos^2\theta + \sin^2\theta) - 2a^2 e\cos\theta + a^2e^2(1 - \sin^2\theta)}$$

$$r = \sqrt{a^2 - 2a^2 e\cos\theta + a^2e^2\cos^2\theta}$$

$$r = \sqrt{(a - ae\cos\theta)^2}$$

$$r = (a - ae\cos\theta)$$

$$\therefore \boxed{r = a(1 - e\cos\theta)}$$

4.2 Equation for elliptical path objects

Now that we know the equation for distance in case of an elliptical orbit, we can obtain the formula for Binding Energy for elliptically revolving objects.

$$E = \frac{1}{2}\tau$$

$$E = \frac{1}{2}\frac{G\,Mm}{r}$$

$$\boxed{E = \frac{1}{2}\frac{G\,Mm}{a(1 - e\cos\theta)}}$$

V BINDING ENERGY FOR EARTH

The Earth revolves around the Sun in an elliptical orbit. This is done in such a way that the Sun is at one of its two foci.

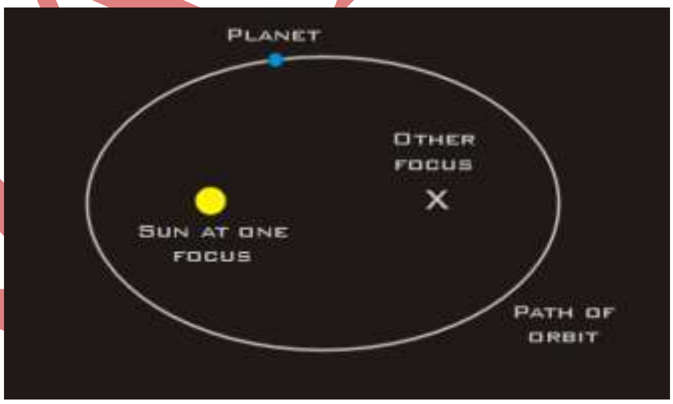

Fig.7. Elliptical motion around the Sun

$$\therefore E = \frac{1}{2}\frac{G\,M_s M_e}{a(1 - e\cos\theta)}$$

$$E = \frac{2.65\times 10^{33}}{(1 - 0.0167\cos\theta)}\ \mathrm{N\cdot m}\ \text{---}\ (1)$$

Where,

$$M_e = Mass\ of\ Earth = 5.9736\times 10^{24}\ kg$$

$M_s = Mass\ of\ the\ Sun = 1.989 \times 10^{30}\ kg$
$a = Semimajor\ axis\ of\ Earth = 149.6 \times 10^{6}\ km$
$G = Gravitational\ Constant = 6.67398 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$
$e = Orbital\ eccentricity\ of\ Earth = 0.0167$

5.1 Binding Energy when Earth is closest to Sun

At this instant the value of $\theta = 0°$. On substituting the value of θ in ①, we get,

$$E = \mathbf{2.695} \times \mathbf{10^{33}}\ \mathrm{N \cdot m}$$

5.2 Binding Energy when Earth is farthest from Sun

At this instant the value of $\theta = 180°$. On substituting the value of θ in ①, we get,

$$E = \mathbf{2.606} \times \mathbf{10^{33}}\ \mathrm{N \cdot m}$$

Similarly, the binding energy of Earth can be easily calculated at any particular instant of its orbital position.

VI BINDING ENERGY FOR COMETS/ASTEROIDS

Similar to planets, most comets revolve around the sun in elliptical orbits. Hence, they also obey the above derived equation

$$E = \frac{1}{2} \frac{G\, M_s M_c}{a(1 - e\cos\theta)}$$

6.1 Halley's Comet

Halley's Comet is one of the most commonly known comets that orbit the Sun. It is visible from the Earth with the naked eye and appears once every 76 years. It appeared most recently in 1986.

$$E = \frac{1}{2} \frac{G\, M_s M_h}{a(1 - e\cos\theta)}$$

Where,

$M_h = Mass\ of\ the\ Halley's\ Comet = 2.2 \times 10^{14}\ kg$

$a = Semimajor\ axis\ of\ the\ Comet = 2.66 \times 10^{9}\ km$ [4]

$e = Orbital\ eccentricity\ of\ the\ Comet = 0.967$

$$\therefore E = \frac{\mathbf{5.489} \times \mathbf{10^{21}}}{(1 - 0.967\ \cos\theta)}\ \mathrm{N \cdot m}$$

Similarly, the Binding Energy of any comet can be obtained for any particular orbital position.

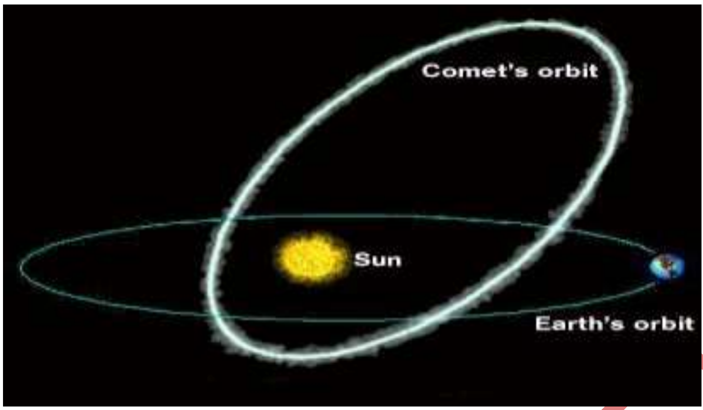

Fig.8. Orbit of Halley's Comet around the Sun

VII CONCLUSION

Now that we have understood how to calculate Binding Energy and discussed some of its applications, we need to understand how significant the knowledge of Binding Energy is. There have been numerous occasions of air bursts and impact events. Many of them had devastating effects. Some notable ones include the Chelyabinsk meteor and the Tunguska event [5]. Even though these events were disastrous, human kind should still consider itself lucky, as any impact by an extraterrestrial body can potentially be extremely catastrophic. NASA has suggested certain courses of action which can be implemented to prevent such incidents [6]. The important thing to be noted is, irrespective of the method we adopt – Pulsed Laser, Mass Driver, Gravity Tractor or Asteroid Tug; it will always be beneficial if we know the binding energy of the Potentially Hazardous Object (PHO).

For instance, if we choose to use nuclear explosives, knowing the binding energy of the PHO would give us a much better idea about the required intensity of the blast. If the PHO is detected early, shifting it even by 5-10 degrees from its trajectory would be enough. In such cases, if the blast provides energy equivalent to 20-30% of the binding energy or even lesser, the PHO could be deviated from its path; successfully preventing a collision with the Earth. Hence, the knowledge of binding energy is crucial.

The Earth has been hit various times by meteors which have the potential of causing enormous loss to both lives and property. It is therefore essential for us to scheme different methods through which we can prevent the Earth from disastrous effects. The equation has been put forth with the intention to help solve this problem.

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